

# Calculus 2

# Choose the method.

# Set it up. Check the result.

Formula, method-selection & common-mistakes guide

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**USE THIS GUIDE IN THREE PASSES**

**Recognize:** match the structure to a method.

**Set up:** write the formula, bounds, and required conditions.

**Verify:** differentiate, check endpoints, or check geometry and units.

| Find it fast                                                             | Pages |
|--------------------------------------------------------------------------|-------|
| Integration: method choice, trig, rational functions, improper integrals | 2-4   |
| Applications: area, work, pressure, volume, length, surface area         | 5-7   |
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A reference for common Calculus II topics, including topics shared with AP Calculus BC. Course coverage varies: use your syllabus to identify required material. This is a study guide, not a complete course or an exam-permitted formula sheet.

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INTEGRATION | RECOGNITION

# Choose an integration method

**FIRST: SIMPLIFY AND CHECK FOR A DERIVATIVE**  
Factor, divide, split, or use an identity. Then look for an inside function and its derivative. A product alone does not make integration by parts the best choice.

| Cue                                                | First move                                                           |
|----------------------------------------------------|----------------------------------------------------------------------|
| Inside function with its derivative                | Substitution: set u equal to the inside function.                    |
| Product; one factor simplifies when differentiated | Integration by parts. LIATE is a hint, not a rule.                   |
| Integer powers of sine/cosine or tangent/secant    | Use a trig identity and a matching substitution (p. 3).              |
| Rational function P(x)/Q(x)                        | Check substitution; divide if needed, then partial fractions (p. 4). |
| Square root of a quadratic                         | Complete the square; consider trig substitution (p. 3).              |
| Infinite bound or unbounded integrand              | Write improper limits before evaluating (p. 4).                      |

## Nearly the same appearance, different first moves

$\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C \quad (u = x^2)$   
 $\int x e^x dx = x e^x - \int e^x dx = (x - 1) e^x + C$   
First integral: the derivative of  $x^2$  is present up to a constant. Second: choose  $u = x$  and  $dv = e^x dx$ . Differentiate either answer to check it.

## Core antiderivatives

$\int x^p dx = \frac{x^{p+1}}{p+1} + C \quad (p \neq -1), \quad \int \frac{dx}{x} = \ln|x| + C$   
 $\int e^{kx} dx = \frac{e^{kx}}{k} + C \quad (k \neq 0), \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$   
 $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$   
For the inverse-trig formulas,  $a > 0$ . Use formulas on intervals where the integrand is defined. For definite substitution, convert the bounds to u or return fully to x before using the original bounds.

INTEGRATION | CONDITIONS MATTER

# Parts and trigonometric patterns

$$\int u \, dv = uv - \int v \, du$$

Choose  $u$  to simplify under differentiation and  $dv$  so it can be integrated. LIATE suggests logarithmic, inverse trig, algebraic, trig, exponential. Repeated polynomial differentiation suggests tabular parts. Cyclic parts may return the original integral; collect it and solve algebraically.

| Integer powers $m, n \geq 0$          | Reserve / substitute                                       |
|---------------------------------------|------------------------------------------------------------|
| $\sin^m x \cos^n x$ : $m$ odd         | Reserve $\sin x$ ; use $u = \cos x$ .                      |
| $\sin^m x \cos^n x$ : $n$ odd         | Reserve $\cos x$ ; use $u = \sin x$ .                      |
| Both sine/cosine powers even          | Use half-angle identities.                                 |
| $\tan^m x \sec^n x$ : $n \geq 2$ even | Reserve $\sec^2 x$ ; use $u = \tan x$ .                    |
| $m$ odd, $m \geq 1$ , and $n \geq 1$  | Reserve $\sec x \tan x$ ; use $u = \sec x$ .               |
| Other tangent/secant cases            | Rewrite with identities; use basic integrals or reduction. |

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \tan^2 x = \sec^2 x - 1$$

## Save exactly the derivative you need

$$\int \tan^3 x \sec x \, dx = \int (u^2 - 1) \, du = \frac{u^3}{3} - u + C \quad (u = \sec x)$$

Return to  $x$ :  $\sec^3 x / 3 - \sec x + C$ . With no secant factor, the same shortcut is unavailable; rewrite  $\tan^3 x = \tan x (\sec^2 x - 1)$ .

## Trig substitution: assume $a > 0$

| Radical            | Substitution        | Differential                                |
|--------------------|---------------------|---------------------------------------------|
| $\sqrt{a^2 - x^2}$ | $x = a \sin \theta$ | $dx = a \cos \theta \, d\theta$             |
| $\sqrt{a^2 + x^2}$ | $x = a \tan \theta$ | $dx = a \sec^2 \theta \, d\theta$           |
| $\sqrt{x^2 - a^2}$ | $x = a \sec \theta$ | $dx = a \sec \theta \tan \theta \, d\theta$ |

Choose a  $\theta$ -interval consistent with the  $x$ -domain. In particular,  $\sqrt{a^2 \tan^2 \theta} = a |\tan \theta|$ . Resolve the sign before removing the absolute value, and return to  $x$  at the end.

## INTEGRATION | SETUP AND LIMITS

# Rational and improper integrals

Partial fractions: long-divide first when degree  $P \geq$  degree  $Q$ . Factor the denominator over the reals. Include every repeated power; use linear numerators over irreducible quadratics.

| Factor in the denominator             | Required contribution                                                        |
|---------------------------------------|------------------------------------------------------------------------------|
| $(x - a)^k$                           | $\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_k}{(x - a)^k}$ |
| $q(x)^k$ ( $q$ irreducible quadratic) | $\frac{A_1x + B_1}{q(x)} + \cdots + \frac{A_kx + B_k}{q(x)^k}$               |

## A complete decomposition

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{dx}{x(x+1)} = \ln|x| - \ln|x+1| + C$$

Valid on intervals avoiding -1 and 0. Recombine the fractions to verify the algebra.

## Improper means a limit is part of the definition

$$\int_a^\infty f(x) dx = \lim_{B \rightarrow \infty} \int_a^B f(x) dx$$

$$\int_a^c f(x) dx = \lim_{t \rightarrow c^-} \int_a^t f(x) dx$$

Split at every interior singularity. For an integral over the entire real line, split at any finite point. Each piece must converge separately; infinite terms cannot cancel.

### TRAP: SYMMETRY DOES NOT REPAIR DIVERGENCE

The integral of  $1/x$  from -1 to 1 diverges: its left-hand piece tends to  $-\infty$  and its right-hand piece tends to  $+\infty$ . The symmetric principal value is 0, but that is a different notion.

$$\int_1^\infty x^{-p} dx \text{ converges} \iff p > 1$$

$$\int_0^1 x^{-p} dx \text{ converges} \iff p < 1$$

For comparison of nonnegative integrands, an upper bound by a convergent integral proves convergence; a lower bound by a divergent integral proves divergence.

APPLICATIONS | BUILD THE QUANTITY

# Area, work, and fluid force

DEFINE THE VARIABLE BEFORE THE FORMULA

Sketch a typical strip. Label its width, height, depth, or lifting distance. Then write “small contribution = rate × small amount” and integrate.

| Quantity                              | Formula and setup                                                                                                                                                                                                                    |
|---------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Area between curves                   | <div><math display="block">A = \int_a^b  f(x) - g(x)  \, dx</math><p>Find intersections; split where top and bottom switch. With dy, use right minus left.</p></div>                                                                 |
| Average value                         | <div><math display="block">f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx</math><p>Assume <math>a &lt; b</math>. This is an average height, not an area.</p></div>                                                               |
| Work along a line                     | <div><math display="block">W = \int_a^b F(x) \, dx</math><p>Use the signed force component along the displacement. For quasistatic stretching from equilibrium, applied spring force is <math>kx</math>.</p></div>                   |
| Pumping fluid                         | <div><math display="block">W = \int \gamma A(y) D(y) \, dy</math><p><math>\gamma</math> = weight per unit volume; <math>A(y)</math> = slice area; <math>D(y)</math> = lift distance. Integrate over the fluid levels.</p></div>      |
| Hydrostatic force on a vertical plate | <div><math display="block">F = \int \gamma h(y) w(y) \, dy</math><p>Gauge pressure: <math>\gamma = \rho g</math>, <math>h</math> = depth below the surface, <math>w</math> = strip width. Integrate over submerged levels.</p></div> |

## Depth is a distance, not a coordinate name

A vertical rectangle is 2 m wide and 3 m tall, with its top at the water surface. Let  $y$  measure depth downward. With water weight density  $\gamma = 9800 \text{ N/m}^3$ :

$$F = 9800 \int_0^3 2y \, dy = 88,200 \text{ N}$$

If  $y$  instead measured height upward from the bottom, depth would be  $3 - y$ . Either coordinate gives the same force.

**Units check:** work = force × distance; pressure = force/area; hydrostatic force = pressure × area. If weight density is supplied, do not multiply by  $g$  again.

APPLICATIONS | COMPARE THE GEOMETRY

# Washers or shells?

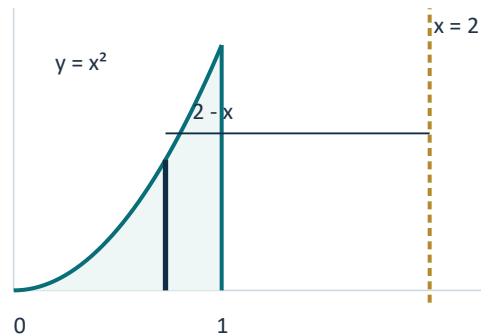
|                 | Washers / disks            | Shells                                    |
|-----------------|----------------------------|-------------------------------------------|
| Slice direction | Perpendicular to the axis  | Parallel to the axis                      |
| Small volume    | $\pi(R^2 - r^2) d\xi$      | $2\pi(\text{radius})(\text{height}) d\xi$ |
| Choose when     | Radii are easy; few splits | Heights are easy; avoids inversion        |

Here  $\xi$  is the slicing coordinate (x or y). About  $x = c$ : shell radius =  $|x - c|$ . About  $y = c$ : shell radius =  $|y - c|$ . Washer radii are distances to the axis, with  $R \geq r \geq 0$ .

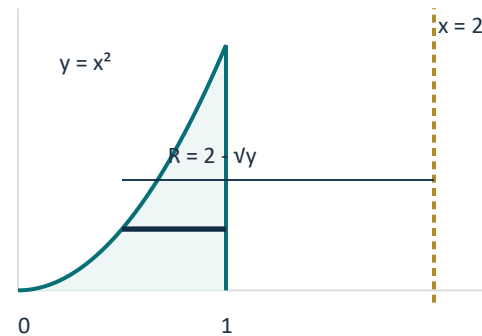
## One region, two correct setups

Rotate the region  $0 \leq x \leq 1, 0 \leq y \leq x^2$  about  $x = 2$ . The axis lies to the right of the region.

Shell: vertical slice



Washer: horizontal slice



$$V = 2\pi \int_0^1 (2 - x)x^2 dx = 2\pi \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5\pi}{6}$$

**Shells:** radius  $2 - x$ , height  $x^2$ , thickness  $dx$ .

$$V = \pi \int_0^1 [(2 - \sqrt{y})^2 - 1^2] dy = \frac{5\pi}{6}$$

**Washers:** at height  $y$ , the region runs from  $x = \sqrt{y}$  to  $x = 1$ . Outer radius is  $2 - \sqrt{y}$ ; inner radius is  $1$ .

WHEN THE REGION CROSSES THE AXIS

If a connected washer slice crosses the axis, its inner radius is 0. Shells from opposite sides can overlap: choose a setup that counts each shell once. Nonnegative radii alone do not prevent double-counting.

**Method choice:** both work here, but shells give the shorter algebra. Choose from the geometry before choosing a formula.

APPLICATIONS | A SHARED BUILDING BLOCK

# Arc length and surface area

$$ds = \sqrt{1 + [f'(x)]^2} \, dx \quad (y = f(x), \, a \leq x \leq b)$$

| Goal                  | Accumulate                  | Units  |
|-----------------------|-----------------------------|--------|
| Arc length            | $L = \int_a^b ds$           | Length |
| Surface of revolution | $S = 2\pi \int_a^b r \, ds$ | Area   |

Assume a sufficiently smooth curve. In the surface formula,  $r$  is the distance to the axis. A geometric surface should be counted once; avoid retracing or overlapping swept pieces.

| Axis of rotation        | Radius for $y = f(x)$ |
|-------------------------|-----------------------|
| x-axis                  | $r =  f(x) $          |
| y-axis                  | $r =  x $             |
| Horizontal line $y = c$ | $r =  f(x) - c $      |
| Vertical line $x = c$   | $r =  x - c $         |

## The extra radius changes the question

For  $y = x$  on  $0 \leq x \leq 1$ , the arc element is  $ds = \sqrt{2} \, dx$ .

$$L = \int_0^1 \sqrt{2} \, dx = \sqrt{2}$$

$$S_{\text{about } x\text{-axis}} = 2\pi \int_0^1 x \sqrt{2} \, dx = \pi \sqrt{2}$$

The second answer is the lateral area of a cone. A closed cone also has a base disk of area  $\pi$ . Add it only when the requested surface includes the base.

## Other representations

$$ds = \sqrt{1 + [g'(y)]^2} \, dy \quad (x = g(y)), \quad ds = \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt$$

CHECK BEFORE EXPANDING

Simplify the expression under the square root. A perfect square may collapse the integral; use  $\sqrt{h^2} = |h|$  and determine the sign on the interval.

## COORDINATES | TANGENTS, AREA, LENGTH

# Parametric and polar curves

## Parametric: $x = x(t)$ , $y = y(t)$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}, \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{x'(t)} \quad (x'(t) \neq 0)$$

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt, \quad \int y dx = \int_{\alpha}^{\beta} y(t)x'(t) dt$$

**Regular tangents:** horizontal when  $y' = 0$  and  $x' \neq 0$ ; vertical when  $x' = 0$  and  $y' \neq 0$ . If both vanish, investigate further. Substitute  $t$  into both coordinates to report a point.

### PARAMETRIC AREA IS SIGNED

The integral  $\int y dx$  tracks orientation and can cancel. For a graph traced once monotonically in  $x$ , geometric area to the  $x$ -axis is  $\int |y(t)| |x'(t)| dt$ . For loops or retracing, analyze the region and split first.

**Quick check:**  $x = 1 - t$ ,  $y = 1$ ,  $0 \leq t \leq 1$  gives  $\int y dx = -1$ , while the geometric area to the  $x$ -axis is 1.

## Polar: $r = r(\theta)$

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta, \quad L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta$$

Area requires a region swept once. Analyze the tracing interval before using symmetry. For two nonnegative radial boundaries on the same rays, subtract inner radius squared from outer radius squared. The slope formula requires a nonzero denominator.

## A full angular turn can be too much

The circle  $r = 2 \cos \theta$  is traced once with  $r \geq 0$  on  $-\pi/2 \leq \theta \leq \pi/2$ .

$$A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \cos^2 \theta d\theta = \pi$$

Using  $0 \leq \theta \leq 2\pi$  traces this circle twice and doubles the area integral.

CONVERGENCE | CHOOSE THE RIGHT OBJECT

# Sequences and series

| Sequence                                                                                       | Series                                                                                         |
|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| A list $a_1, a_2, \dots$ ; ask whether $a_n$ approaches a finite limit.                        | A sum $\sum a_n$ ; ask whether its partial sums approach a finite limit.                       |
| An increasing sequence bounded above converges. A decreasing sequence bounded below converges. | Define $S_N = a_1 + \dots + a_N$ . The series converges exactly when $S_N$ has a finite limit. |

**STEP ZERO: THE TERMS MUST TEND TO ZERO**  
If  $a_n$  does not tend to 0, then  $\sum a_n$  diverges. If  $a_n$  tends to 0, this test says nothing more. The harmonic series  $\sum 1/n$  is the standard warning.

## Read the structure, then choose a test

| Structure                                      | Natural first test                                              |
|------------------------------------------------|-----------------------------------------------------------------|
| Constant ratio or a known power of $n$         | Geometric or $p$ -series benchmark                              |
| Differences that cancel in a partial sum       | Telescoping; write $S_N$ before taking a limit                  |
| Positive terms resembling a benchmark          | Direct comparison or limit comparison                           |
| Positive terms $f(n)$ with manageable integral | Integral test; check eventual decrease                          |
| Factorials, exponentials, or products          | Ratio test                                                      |
| An expression raised to the $n$ th power       | Root test                                                       |
| Alternating signs                              | Check absolute convergence and/or alternating-series conditions |

## Telescoping gives the sum, not just convergence

$$S_N = \sum_{n=1}^N \frac{1}{n(n+1)} = \sum_{n=1}^N \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$$
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Write enough terms to show the cancellation and retain the final uncanceled term.

**Tail principle:** adding or removing finitely many terms does not change convergence, although it can change the sum. Most sign and monotonicity conditions only need to hold eventually.

CONVERGENCE | HYPOTHESES AND CONCLUSIONS

# Series tests at a glance

| Test              | What to check                                                                                               | What follows                                                                                                  |
|-------------------|-------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------|
| Geometric         | $\sum_{n=0}^{\infty} ar^n$                                                                                  | For $ r  < 1$ , sum = $a/(1 - r)$ . For $ r  \geq 1$ and $a \neq 0$ , diverges. The starting index matters.   |
| p-series          | $\sum_{n=1}^{\infty} \frac{1}{n^p}$                                                                         | Converges exactly when $p > 1$ .                                                                              |
| Direct comparison | Eventually $0 \leq a_n \leq b_n$ .                                                                          | If $\sum b_n$ converges, then $\sum a_n$ converges.<br>If $\sum a_n$ diverges, then $\sum b_n$ diverges.      |
| Limit comparison  | $a_n, b_n > 0$ eventually; $\lim a_n/b_n = L$ .                                                             | If $0 < L < \infty$ , both series have the same convergence behavior.                                         |
| Integral          | $a_n = f(n)$ ; $f$ positive, continuous, decreasing on a tail.                                              | $\sum a_n$ and the corresponding improper integral converge or diverge together; their values need not agree. |
| Alternating       | Terms are $(-1)^n b_n$ or $(-1)^{n+1} b_n$ ; $b_n \geq 0$ , eventually nonincreasing, $b_n \rightarrow 0$ . | The series converges. Check $\sum b_n$ separately to classify absolute or conditional convergence.            |
| Ratio             | $L = \lim_{n \rightarrow \infty} \left  \frac{a_{n+1}}{a_n} \right $                                        | $L < 1$ : absolute convergence.<br>$L > 1$ (including $\infty$ ): divergence.<br>$L = 1$ : inconclusive.      |
| Root              | $L = \lim_{n \rightarrow \infty} \sqrt[n]{ a_n }$                                                           | $L < 1$ : absolute convergence.<br>$L > 1$ (including $\infty$ ): divergence.<br>$L = 1$ : inconclusive.      |

For these limit forms, the limit must exist (possibly  $\infty$ ). The ratio also requires nonzero denominator terms eventually. If these conditions fail, choose another test.

**ABSOLUTE OR CONDITIONAL?**

**Absolute:**  $\sum |a_n|$  converges, so  $\sum a_n$  converges.

**Conditional:**  $\sum a_n$  converges but  $\sum |a_n|$  diverges.

**Comparison direction:** smaller than a divergent series proves nothing; larger than a convergent series proves nothing. For a rational function of  $n$ , compare its leading powers.

## POWER SERIES | RADIUS IS ONLY STEP ONE

## Find the interval of convergence

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

1. Identify the center  $a$ . 2. Apply ratio/root to absolute values. 3. Solve for  $|x-a| < R$ . 4. If  $0 < R < \infty$ , test both endpoints in the original series. 5. State radius, interval, and endpoint behavior.

**SPECIAL RADII**

$R = 0$ : only the center converges.

$R = \infty$ : every real  $x$  converges.

For  $0 < R < \infty$ , convergence is absolute inside and fails outside; endpoints require separate tests.

## Worked example: one endpoint included

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-2|}{3} \frac{n}{n+1} \longrightarrow \frac{|x-2|}{3}$$

For  $x \neq 2$ , the ratio is less than 1 in the limit when  $|x-2| < 3$ . At the center  $x = 2$ , every term is zero. Thus  $R = 3$  and the open interval is  $(-1, 5)$ .

| Endpoint | Substitute into the original series    | Result                                         |
|----------|----------------------------------------|------------------------------------------------|
| $x = -1$ | $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ | Converges conditionally: alternating harmonic. |
| $x = 5$  | $\sum_{n=1}^{\infty} \frac{1}{n}$      | Diverges: harmonic series.                     |

$$R = 3, \quad \text{interval} = [-1, 5)$$

## Substitution, differentiation, and integration

Transform a known power series instead of repeatedly differentiating a function. Termwise differentiation and integration are valid inside the radius and preserve that radius. They may change endpoint convergence, so test endpoints again.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad (|x| < 1)$$

This comes from the geometric series with ratio  $-x^2$ . At  $x = \pm 1$  the terms do not approach zero, so neither endpoint is included.

Taylor Series | Representation and Reuse

# Taylor and Maclaurin essentials

$$T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n, \qquad R_N(x) = f(x) - T_N(x)$$

The Taylor series is the corresponding infinite series. It represents  $f(x)$  where  $R_N(x) \rightarrow 0$ ; having all derivatives is not enough. Maclaurin means  $a = 0$ . Keep the derivative, factorial, and power on the same index.

| Function        | Maclaurin series                                      | Real interval |
|-----------------|-------------------------------------------------------|---------------|
| $e^x$           | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$                  | All real $x$  |
| $\sin x$        | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ | All real $x$  |
| $\cos x$        | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$     | All real $x$  |
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n$                             | $(-1, 1)$     |
| $\ln(1+x)$      | $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$        | $(-1, 1]$     |
| $\arctan x$     | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$    | $[-1, 1]$     |

The logarithm series at  $x = 1$  and arctangent series at  $x = \pm 1$  converge conditionally. The geometric, logarithm, and arctangent series converge absolutely for  $|x| < 1$ .

## Build a new series from one you know

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

The interval remains all real  $x$ . The degree-6 polynomial shown has four nonzero terms; degree and number of terms are different counts.

## APPROXIMATION | MAKE ACCURACY A PROOF

# Error bounds you can justify

## Alternating-series remainder

$$S_N = \sum_{n=1}^N (-1)^{n+1} b_n, \quad |S - S_N| \leq b_{N+1}$$

Use when the omitted tail alternates, its nonnegative magnitudes are nonincreasing, and they tend to 0. The first omitted term controls the error; the remainder has that term's sign.

## How many terms for $\ln(1.2)$ within $10^{-4}$ ?

$$\ln(1.2) = \frac{1}{5} - \frac{1}{2 \cdot 5^2} + \frac{1}{3 \cdot 5^3} - \frac{1}{4 \cdot 5^4} + \dots$$

Here  $b_n = 1/(n5^n)$  decreases to 0. After three terms, the next-term bound is  $1/2500 = 0.0004$ , which is too large.

$$S_4 = \frac{1}{5} - \frac{1}{50} + \frac{1}{375} - \frac{1}{2500} = \frac{1367}{7500} \approx 0.182266667$$

$$|S - S_4| \leq \frac{1}{15625} = 0.000064 < 10^{-4}$$

Four terms are the fewest certified by this bound. A failed upper bound does not by itself prove the actual error is too large.

## Taylor / Lagrange remainder

$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!}$$

A sufficient hypothesis:  $f$  has continuous derivatives through order  $N+1$  on the segment joining  $a$  and  $x$ , and  $|f^{(N+1)}(z)| \leq M$  throughout that segment.  $M$  must be a valid uniform bound, not merely a derivative value at one point.

## A derivative bound for $e^{0.2}$

Use the degree-3 Maclaurin polynomial. On  $[0, 0.2]$ , every derivative is  $e^z$ , so the simple bound  $M = 2$  is valid.

$$T_3(0.2) = 1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{6} = \frac{458}{375}$$

$$|R_3(0.2)| \leq \frac{2(0.2)^4}{4!} = \frac{1}{7500} < 0.000134$$

**Accuracy workflow:** select a valid bound, solve the tolerance inequality, check the chosen index, and state whether the request was for degree or number of terms.

## APPROXIMATION | ADDITIONAL REFERENCE

# Numerical integrals and series tails

Use these tools when your course includes numerical integration or asks for a quantitative tail estimate. Equal spacing is required for the composite rules below.

$$h = \frac{b-a}{n}, \quad x_i = a + ih, \quad f_i = f(x_i)$$

## Composite trapezoidal rule

$$T_n = \frac{h}{2} \left[ f_0 + 2 \sum_{i=1}^{n-1} f_i + f_n \right]$$

$$|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2}, \quad I = \int_a^b f(x) dx$$

Require  $a < b$ ,  $n \geq 1$ , and continuous  $f''$  with  $|f''(x)| \leq K_2$  on  $[a, b]$  for this error bound.

## Composite Simpson's rule

$$S_n = \frac{h}{3} [f_0 + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2}) + f_n]$$

$$|I - S_n| \leq \frac{K_4(b-a)^5}{180n^4}$$

$n$  must be positive and even. For this error bound, require continuous  $f^{(4)}$  with  $|f^{(4)}(x)| \leq K_4$  on  $[a, b]$ . Weights are 1, 4, 2, 4, ..., 2, 4, 1.

### SUBINTERVALS ARE NOT SAMPLE POINTS

$n$  subintervals use  $n + 1$  sample points. If an error inequality gives a noninteger threshold for  $n$ , round up; for Simpson's rule, then choose an even  $n$ .

## Positive-series tail: the integral test gives bounds

$$\int_{N+1}^{\infty} f(x) dx \leq R_N = \sum_{n=N+1}^{\infty} f(n) \leq \int_N^{\infty} f(x) dx$$

Require  $f$  positive, continuous, nonincreasing on  $[N, \infty)$ , and a convergent tail integral. These inequalities bound the error after the first  $N$  terms.

$$\frac{1}{N+1} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \frac{1}{N}$$

For  $\Sigma 1/n^2$ , taking  $N = 100$  guarantees tail error  $\leq 0.01$  by this bound. Add the bounds to the partial sum to enclose the full sum.

## ACTIVE RECALL | COVER THE ANSWERS

## Six quick checks

For each prompt, name the method and write the first justified line before calculating. Then compare with the answers below.

| Prompt                                                                                       | Your first line |
|----------------------------------------------------------------------------------------------|-----------------|
| 1 $\int x \cos(x^2) dx$                                                                      | _____           |
| 2 $\int_0^1 x^{-1/2} dx$                                                                     | _____           |
| 3 Rotate $0 \leq y \leq x$ , $0 \leq x \leq 1$ about the y-axis. Set up shells.              | _____           |
| 4 $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$                                                    | _____           |
| 5 $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$                                                 | _____           |
| 6 Use $1 - x^2/2$ to approximate $\cos(0.1)$ . Bound the error using the alternating series. | _____           |

## Answers and the reason they work

1. **Substitution:**  $u = x^2$ ,  $du = 2x dx$ . Answer:  $(1/2)\sin(x^2) + C$ . Differentiate to check.

2. **Improper at 0:** take the limit as  $\epsilon \rightarrow 0^+$  of the integral from  $\epsilon$  to 1. Its value is  $\lim(2 - 2\sqrt{\epsilon}) = 2$ .

$$3. \quad V = 2\pi \int_0^1 x \cdot x dx = \frac{2\pi}{3}$$

4. **Limit comparison:** compare with  $1/n^2$ . The ratio is  $n^3/(n^3 + 1) \rightarrow 1$ . Both positive-term series converge.

5. **Ratio test:**  $R = 2$  and open interval  $(-1, 3)$ . At -1 the series is alternating harmonic; at 3 it is harmonic. Interval: **[-1, 3)**.

6. **Alternating remainder:** approximation = 0.995. The terms decrease to 0 at  $x = 0.1$ , and the first omitted term is positive:

$$0 \leq \cos(0.1) - 0.995 \leq \frac{(0.1)^4}{4!} = \frac{1}{240000} \approx 0.000004167$$

## IF YOU MISSED ONE

1: derivative recognition (p. 2). 2: improper limits (p. 4). 3: shell geometry (p. 6). 4: comparison (pp. 9-10). 5: endpoint tests (p. 11). 6: first omitted term (p. 13).

FINAL VERIFICATION | KEEP THE POINTS

# Common mistakes → useful repairs

| Instead of this mistake...                       | Write or check this...                                     |
|--------------------------------------------------|------------------------------------------------------------|
| Choose a method from appearance alone            | Simplify; look for a derivative match first.               |
| Mix x-bounds with a u-integral                   | Change the bounds, or return fully to x.                   |
| Forget +C or logarithm absolute values           | Include +C for indefinite integrals; use $\ln x $ .        |
| Omit a repeated factor in partial fractions      | Include every denominator power.                           |
| Cancel divergent improper pieces                 | Split; require each improper piece to converge.            |
| Use a coordinate as radius, depth, or lift       | Label the actual distance to the axis, surface, or outlet. |
| Confuse length with surface area                 | Length uses ds; surface area uses $2\pi r$ ds.             |
| Conclude convergence because $a_n \rightarrow 0$ | Use another test; this condition is only necessary.        |
| Treat ratio/root $L = 1$ as an answer            | The test is inconclusive; select a different test.         |
| Stop after finding a power-series radius         | Test each finite endpoint in the original series.          |
| Assume a Taylor series equals the function       | Justify that the remainder tends to 0.                     |
| Use a bound without its hypotheses               | Check monotonicity or the uniform derivative bound.        |
| Report only t, or trace a polar curve twice      | Convert to (x, y); verify the tracing interval.            |

FINAL 60-SECOND SCAN

Did I answer the exact question? Are bounds, signs, and endpoints correct? Are distances and units sensible? Did I state convergence type, interval notation, or point coordinates as requested? Can I differentiate or estimate to check the result?

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