Dickson Trace Curves and Asymptotic Sparsity for Reciprocal Quadrinomials over $\mathbb{F}_{q^2}$

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Abstract

We study a coefficient-linked reciprocal family of sparse polynomials over $\mathbb{F}_{q^2}$. The reciprocal structure gives a universal collision identity: outside the natural degenerate cases $b = 0, \pm 1$ and $a = \pm b$, all off-diagonal collisions of the induced rational map on the unit circle are independent of the parameter a.

For odd degrees $n = 2m + 1$, the collision equation reduces to a Dickson trace curve

$$\mathcal{C}_{n,b}: \quad bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

For admissible trace curves, a quadratic-character sum argument gives the explicit lower bound

$$N_{n,b} \geq \frac{q - 2(n-2)^2\sqrt{q} - (20n - 19)}{8}$$

for the number of strict lifted collisions. Thus, when this lower bound is positive, no nondegenerate member of the corresponding b-fiber can be a permutation polynomial.

Under the stated characteristic hypothesis, we construct an effective exceptional polynomial $E_{n,q}(b)$ whose nonvanishing guarantees admissibility. Consequently, for each fixed odd n, possible nondegenerate permutation examples in this reciprocal quadrinomial family are confined to $O_n(1)$ vertical b-fibers and have density $O_n(1/q)$ in the full nondegenerate parameter space.

We compute two explicit higher-degree cases. For $n = 7$, the trace curve is cubic and gives a generic non-permutation theorem outside an explicit exceptional polynomial $E_7(b)$, with large-field candidate-density bound $10/(q-3)$. For $n = 9$, the trace curve is quartic and gives a generic non-permutation theorem outside an explicit exceptional polynomial $E_9(b)$, with large-field candidate-density bound $14/(q-3)$. The nondegenerate exceptional degrees 10 and 14 match $2n - 4$ in these first two higher-degree cases, suggesting a possible linear exceptional-degree pattern for future work.

Contents

1 Introduction

Permutation polynomials over finite fields are classical objects with connections to finite geometry, coding theory, cryptography, and combinatorial design theory. Sparse permutation polynomials over quadratic extensions $\mathbb{F}_{q^2}$ are often studied through root-of-unity reductions. In this setting, a polynomial of the form

$$x^r h(x^{q-1})$$

is reduced to a coprimality condition and a rational-map permutation problem on the unit circle

$$\mu_{q+1} = \{z \in \mathbb{F}_{q^2} : z^{q+1} = 1\}.$$

This approach is closely related to the Akbary–Ghioca–Wang criterion [1] and has been used extensively in the study of sparse permutation polynomials over $\mathbb{F}_{q^2}$; see, for example, [6, 2].

This paper studies the reciprocal quadrinomial family

$$F_{n,a,b}(x) = x^n + ax^{q+n-1} + bx^{(n-1)q+1} + \frac{a}{b}x^{nq}, \qquad a, b \in \mathbb{F}_q.$$

Equivalently,

$$F_{n,a,b}(x) = x^n h_{n,a,b}(x^{q-1}),$$

where

$$h_{n,a,b}(z) = 1 + az + bz^{n-1} + \frac{a}{b}z^n.$$

The reciprocal polynomial is

$$h^*_{n,a,b}(z) = z^n h_{n,a,b}(1/z) = z^n + az^{n-1} + bz + \frac{a}{b}.$$

On reciprocal polynomial is

$$h^*_{n,a,b}(z) = z^n h_{n,a,b}(1/z) = z^n + \mu_{q+1}, theinduced rational map is R_{n,a,b}(z) = \frac{h^*_{n,a,b}(z)}{h_{n,a,b}(z)}.$$

The case $n = 5$ is the reciprocal degree-five quadrinomial family treated in the companion classification paper [11]. In that case, the trace curve is a conic and the exceptional fibers can be analyzed completely, yielding explicit infinite families and sporadic cases. The present paper shows that this conic is not an isolated phenomenon: it is the $m = 2$ member of a general Dickson trace-curve hierarchy. Beginning at $n = 7$, however, the trace curves have positive genus in the smooth projective case, and the appropriate general conclusion becomes asymptotic sparsity rather than complete classification.

The first main result is a universal collision factorization. Outside the natural degenerate cases

$$b = 0, \qquad b = \pm 1, \qquad a = \pm b,$$

the off-diagonal collision equation for $R_{n,a,b}$ is independent of the parameter a. Thus, in the nondegenerate range, unit-circle injectivity is controlled by a one-parameter family indexed by b, even though the original family has two coefficients a, b.

For odd $n = 2m + 1$, this collision equation admits a trace reduction. If

$$u = xy, \qquad v = \frac{x}{y},$$

and

$$W = u + u^{-1}, \qquad V = v + v^{-1},$$

then the off-diagonal collision equation becomes the Dickson trace curve

$$\mathcal{C}_{n,b} : \quad bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0,$$

where

$$D_j(t + t^{-1}) = t^j + t^{-j},$$

and

$$S_m(X) = 1 + \sum_{j=1}^{m} D_j(X).$$

The first cases form a natural hierarchy: a line for $n = 3$, the conic underlying the degree-five classification for $n = 5$, a cubic for $n = 7$, and a quartic for $n = 9$.

The general theorem proves that admissible trace curves force strict lifted collisions for sufficiently large fields. More precisely, for admissible b, the number $N_{n,b}$ of strict lifted trace points satisfies

$$N_{n,b} \geq \frac{q - 2(n-2)^2\sqrt{q} - (20n - 19)}{8}.$$

When the right side is positive, the induced rational map on μ_{q+1} has an off-diagonal collision, and hence the corresponding reciprocal quadrinomial cannot be a permutation polynomial of $\mathbb{F}_{q^2}$.

Admissibility is made effective by constructing an exceptional polynomial $E_{n,q}(b)$. Nonvanishing of $E_{n,q}(b)$, together with the stated characteristic hypothesis, guarantees admissibility.

It is important to emphasize that the exceptional polynomial $E_{n,q}(b)$ is not claimed to be minimal, nor are its roots claimed to produce permutation polynomials. The role of $E_{n,q}$ is one-sided: if $E_{n,q}(b) \neq 0$, then the trace curve is admissible and the character-sum argument forces an off-diagonal collision. Thus possible permutation examples are confined to the exceptional fibers $E_{n,q}(b) = 0$. Determining which exceptional fibers, if any, contain genuine permutation families is a separate arithmetic problem.

Consequently, for each fixed odd n, possible nondegenerate permutation examples are confined to boundedly many vertical b-fibers. This gives an asymptotic sparsity theorem: the density of possible permutation pairs in the nondegenerate (a, b)-parameter space is $O_n(1/q)$.

The paper also computes two explicit higher-degree cases. For $n = 7$, the trace curve is a cubic and yields an explicit exceptional polynomial $E_7(b)$. If

$$\operatorname{char} \mathbb{F}_q \neq 3, \qquad E_7(b) \neq 0, \qquad q > 3000,$$

then no nondegenerate a gives a permutation polynomial. Inside the nondegenerate parameter space, possible permutation pairs lie over at most ten b-fibers, giving density at most $10/(q - 3)$.

For $n = 9$, the trace curve is a quartic and yields an explicit exceptional polynomial $E_9(b)$. If

$$E_9(b) \neq 0, \qquad q > 10000,$$

then no nondegenerate a gives a permutation polynomial. In this case, possible permutation pairs lie over at most fourteen b-fibers, giving density at most $14/(q - 3)$.

These explicit degrees are much smaller than the general discriminant-degree bound. After removing the universal degenerate factor $b(b^2 - 1)$, the nondegenerate exceptional degrees in the first two higher-degree cases are

$$10 = 2(7) - 4 \qquad \text{and} \qquad 14 = 2(9) - 4.$$

This suggests a possible linear exceptional-degree pattern, although no such general theorem is proved here. The next natural test case is $n = 11$, where the trace curve becomes a quintic.

Main results

The main results of the paper are as follows.

(1) We prove a universal common-root obstruction and collision factorization for the reciprocal quadrinomial family $F_{n,a,b}$. Outside the degenerate cases $b = 0, \pm 1$ and $a = \pm b$, all off-diagonal collisions are governed by a polynomial $K_{n,b}(x, y)$ independent of a.

(2) For odd $n = 2m + 1$, we reduce the collision equation to the Dickson trace curve

$$\mathcal{C}_{n,b} : \quad bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

(3) For admissible trace curves, we prove the lower bound

$$N_{n,b} \geq \frac{q - 2(n-2)^2\sqrt{q} - (20n - 19)}{8}.$$

(4) We define an effective exceptional polynomial $E_{n,q}(b)$. Outside the locus $E_{n,q}(b) = 0$, all sufficiently large fields force strict lifted collisions. Hence possible nondegenerate permutation pairs have density $O_n(1/q)$.

(5) We compute explicit exceptional polynomials for the first two genuinely higher-degree cases $n = 7$ and $n = 9$, obtaining cubic and quartic trace-curve generic non-permutation theorems with density bounds $10/(q-3)$ and $14/(q-3)$, respectively.

2 Root-of-unity reduction and the reciprocal family

Throughout, q is an odd prime power. We write χ for the quadratic character of $\mathbb{F}_q$, extended by $\chi(0) = 0$, and

$$\mu_{q+1} = \{z \in \mathbb{F}_{q^2} : z^{q+1} = 1\}.$$

For $z \in \mu_{q+1}$, one has $z^q = z^{-1}$. For standard background on finite fields, characters, and Dickson polynomials, see Lidl and Niederreiter [7].

Lemma 2.1 (Root-of-unity reduction)**.** *Let $h \in \mathbb{F}_{q^2}[x]$, and let r be a positive integer. Then*

$$f(x) = x^r h(x^{q-1})$$

permutes $\mathbb{F}_{q^2}$ if and only if

$$\gcd(r, q-1) = 1$$

and the map

$$z \longmapsto z^r h(z)^{q-1}$$

permutes μ_{q+1}.

Remark 2.2. This is the standard root-of-unity reduction, often viewed as a multiplicative form of the AGW criterion [1]. In the present paper we use it with $r = n$, so the full-field coprimality condition is $\gcd(n, q-1) = 1$. However, the non-permutation results below are stronger: they produce off-diagonal collisions on μ_{q+1}, and therefore rule out permutation behavior regardless of the coprimality condition.

For $a, b \in \mathbb{F}_q$, with $b \neq 0$, define

$$F_{n,a,b}(x) = x^n + ax^{q+n-1} + bx^{(n-1)q+1} + \frac{a}{b}x^{nq}.$$

Equivalently,

$$F_{n,a,b}(x) = x^n h_{n,a,b}(x^{q-1}),$$

where

$$h_{n,a,b}(z) = 1 + az + bz^{n-1} + \frac{a}{b}z^n.$$

The reciprocal polynomial is

$$h^*_{n,a,b}(z) = z^n h_{n,a,b}(1/z) = z^n + az^{n-1} + bz + \frac{a}{b}.$$

For $z \in \mu_{q+1}$, since $z^q = z^{-1}$ and $a, b \in \mathbb{F}_q$, one obtains

$$z^n h_{n,a,b}(z)^{q-1} = z^n \frac{h_{n,a,b}(z^{-1})}{h_{n,a,b}(z)} = \frac{h^*_{n,a,b}(z)}{h_{n,a,b}(z)}.$$

Thus the induced rational function on μ_{q+1} is

$$R_{n,a,b}(z) = \frac{h^*_{n,a,b}(z)}{h_{n,a,b}(z)}.$$

The natural nondegenerate range in this paper is

$$b \neq 0, \qquad b \neq \pm 1, \qquad a \neq \pm b.$$

The values $a = \pm b$ collapse the induced rational map, while $b = \pm 1$ produces reciprocal cancellation. The next section shows that these are also the universal common-root obstructions.

3 Common roots and universal collision factorization

We first record a common-root obstruction valid in all degrees.

Proposition 3.1 (Universal common-root obstruction)**.** *Let*

$$h(z) = 1 + az + bz^{n-1} + \frac{a}{b}z^n, \qquad h^*(z) = z^n + az^{n-1} + bz + \frac{a}{b}.$$

After clearing denominators, if h and h^* have a common root, then

$$(a-b)(a+b)(b-1)(b+1) = 0.$$

In particular, if

$$b \neq 0, \qquad b \neq \pm 1, \qquad a \neq \pm b,$$

then h and h^* have no common root, and $R_{n,a,b}$ is defined on all of μ_{q+1}.

Proof. Clear denominators by setting

$$H(z) = bh(z) = b + abz + b^2 z^{n-1} + az^n,$$

and

$$G(z) = bh^*(z) = bz^n + abz^{n-1} + b^2 z + a.$$

A direct computation gives

$$aH - bG = (a^2 - b^2)z(b + z^{n-1}),$$

and

$$bH - aG = (b^2 - a^2)(1 + bz^{n-1}).$$

If H and G have a common root z and $a^2 \neq b^2$, then

$$z^{n-1} = -b \qquad \text{and} \qquad z^{n-1} = -\frac{1}{b}.$$

Thus $b^2 = 1$. Hence any common root forces

$$a = \pm b \qquad \text{or} \qquad b = \pm 1.$$

Finally, if $z \in \mu_{q+1}$ and $h(z) = 0$, then $h(z^{-1}) = h(z)^q = 0$, so $h^*(z) = 0$. Therefore, outside the displayed locus, h has no zero on μ_{q+1}, and $R_{n,a,b}$ is defined on the whole unit circle. □

We next factor the off-diagonal collision equation.

Proposition 3.2 (Universal collision factorization)**.** *Let*

$$h(z) = 1 + az + bz^{n-1} + \frac{a}{b}z^n, \qquad h^*(z) = z^n + az^{n-1} + bz + \frac{a}{b}.$$

Set

$$\mathcal{N}_n(x,y) = h^*(x)h(y) - h^*(y)h(x).$$

Then

$$\mathcal{N}_n(x,y) = -\frac{(a-b)(a+b)}{b^2}(x-y)K_{n,b}(x,y),$$

where

$$K_{n,b}(x,y) = \sum_{i=0}^{n-1} x^{n-1-i}y^i + b(1 + x^{n-1}y^{n-1}) - b^2 \sum_{i=1}^{n-2} x^{n-1-i}y^i.$$

Proof. Write $c = a/b$. Then

$$h(z) = 1 + az + bz^{n-1} + cz^n,$$

and

$$h^*(z) = z^n + az^{n-1} + bz + c.$$

The polynomial

$$\mathcal{N}_n(x,y) = h^*(x)h(y) - h^*(y)h(x)$$

is antisymmetric in x, y, hence divisible by $x - y$. Expanding $\mathcal{N}_n(x,y)$, grouping antisymmetric monomial pairs, and using

$$X^rY^s - X^sY^r = (X-Y)X^sY^s \sum_{j=0}^{r-s-1} X^{r-s-1-j}Y^j \qquad (r > s),$$

one obtains

$$\mathcal{N}_n(x,y) = -\frac{(a-b)(a+b)}{b^2}(x-y)K_{n,b}(x,y).$$

The identity is a polynomial identity after clearing denominators. □

Corollary 3.3 (a-independence of off-diagonal collisions)**.** *Assume*

$$b \neq 0, \qquad b \neq \pm 1, \qquad a \neq \pm b.$$

For $x, y \in \mu_{q+1}$ with $x \neq y$,

$$R_{n,a,b}(x) = R_{n,a,b}(y)$$

if and only if

$$K_{n,b}(x,y) = 0.$$

Thus, in the nondegenerate range, off-diagonal collision behavior on μ_{q+1} depends only on b, not on a.

Proof. Under the stated assumptions, Proposition 3.1 shows that $R_{n,a,b}$ is defined on all of μ_{q+1}, and the scalar

$$-\frac{(a-b)(a+b)}{b^2}$$

in Proposition 3.2 is nonzero. Hence, for $x \neq y$,

$$R_{n,a,b}(x) = R_{n,a,b}(y)$$

is equivalent to

$$K_{n,b}(x, y) = 0.$$

□

4 Odd-degree Dickson trace reduction

We now specialize to odd degree

$$n = 2m + 1.$$

Let $D_j(X)$ denote the Dickson polynomial normalized by

$$D_j(t + t^{-1}) = t^j + t^{-j}.$$

Thus

$$D_0(X) = 2, \qquad D_1(X) = X,$$

and

$$D_j(X) = XD_{j-1}(X) - D_{j-2}(X).$$

Define

$$S_m(X) = 1 + \sum_{j=1}^{m} D_j(X).$$

Let

$$u = xy, \qquad v = \frac{x}{y}.$$

For $x, y \in \mu_{q+1}$, one has $u, v \in \mu_{q+1}$. Define the traces

$$W = u + u^{-1}, \qquad V = v + v^{-1}.$$

Theorem 4.1 (Dickson trace curve reduction)**.** *Let $n = 2m + 1$. For $x, y \in \mu_{q+1}$, the equation*

$$K_{n,b}(x, y) = 0$$

is equivalent to

$$bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

Thus the off-diagonal collision equation reduces to the Dickson trace curve

$$\mathcal{C}_{n,b}: \quad bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

Proof. Divide $K_{n,b}(x, y)$ by $(xy)^m = u^m$. Since $n - 1 = 2m$, the first sum becomes

$$\frac{1}{u^m}\sum_{i=0}^{2m} x^{2m-i}y^i = \sum_{i=0}^{2m} v^{m-i} = 1 + \sum_{j=1}^{m}(v^j + v^{-j}) = S_m(V).$$

The middle term becomes

$$\frac{b(1 + x^{2m}y^{2m})}{u^m} = b(u^{-m} + u^m) = bD_m(W).$$

Finally,

$$\frac{1}{u^m}\sum_{i=1}^{2m-1} x^{2m-i}y^i = \sum_{i=1}^{2m-1} v^{m-i} = 1 + \sum_{j=1}^{m-1}(v^j + v^{-j}) = S_{m-1}(V).$$

Therefore

$$\frac{K_{n,b}(x, y)}{(xy)^m} = bD_m(W) + S_m(V) - b^2 S_{m-1}(V).$$

Since $xy \neq 0$, the result follows. □

5 Strict lifted collisions and admissibility

The trace curve $\mathcal{C}_{n,b}$ records trace-level collisions. We now state character conditions ensuring that a trace point lifts to an actual off-diagonal collision on the unit circle.

Let

$$f_1 = W^2 - 4, \qquad f_2 = V^2 - 4, \qquad f_3 = (W + 2)(V + 2).$$

Lemma 5.1 (Strict trace lifting)**.** *Let $W, V \in \mathbb{F}_q$ satisfy*

$$\chi(W^2 - 4) = -1, \qquad \chi(V^2 - 4) = -1, \qquad \chi((W + 2)(V + 2)) = 1.$$

Then there exist $u, v \in \mu_{q+1}$ and distinct $x, y \in \mu_{q+1}$ such that

$$W = u + u^{-1}, \qquad V = v + v^{-1},$$

and

$$xy = u, \qquad \frac{x}{y} = v.$$

Consequently, if $(W, V) \in \mathcal{C}_{n,b}(\mathbb{F}_q)$, then $K_{n,b}(x, y) = 0$, and hence $R_{n,a,b}$ has an off-diagonal collision on μ_{q+1} for every nondegenerate a.

Proof. The equation $T = t + t^{-1}$ is equivalent to

$$t^2 - Tt + 1 = 0.$$

If $\chi(T^2 - 4) = -1$, then the roots lie in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$, have product 1, and satisfy $t^q = t^{-1}$. Hence they lie in μ_{q+1}. Thus the first two character conditions give $u, v \in \mu_{q+1}$ with

$$W = u + u^{-1}, \qquad V = v + v^{-1}.$$

Moreover $W \neq \pm 2$ and $V \neq \pm 2$, so $u, v \neq \pm 1$.

Let η denote the square character on μ_{q+1}. For $t \in \mu_{q+1}$, $t \neq -1$, and $T = t + t^{-1}$, we have

$$T + 2 = \frac{(t+1)^2}{t}.$$

Since $(t+1)^q = (t+1)/t$, it follows that

$$(t+1)^{q-1} = t^{-1}.$$

Therefore

$$\chi(T+2) = \left(\frac{(t+1)^2}{t}\right)^{(q-1)/2} = t^{-(q+1)/2} = \eta(t).$$

Hence

$$\chi(W+2) = \eta(u), \qquad \chi(V+2) = \eta(v).$$

The condition

$$\chi((W+2)(V+2)) = 1$$

therefore implies

$$\eta(u)\eta(v) = 1.$$

Thus uv is a square in μ_{q+1}. Choose $x \in \mu_{q+1}$ with

$$x^2 = uv,$$

and set

$$y = \frac{u}{x}.$$

Then $xy = u$, and

$$\frac{x}{y} = \frac{x^2}{u} = v.$$

Since $V \neq 2$, we have $v \neq 1$, so $x \neq y$.

If (W, V) lies on $\mathcal{C}_{n,b}$, then Theorem 4.1 gives $K_{n,b}(x, y) = 0$. By Corollary 3.3, this gives an off-diagonal collision of $R_{n,a,b}$ on μ_{q+1}. □

We now introduce the geometric hypotheses that make the character sums nontrivial.

Definition 5.2 (Admissibility)**.** Let $X_{n,b}$ be the projective closure of $\mathcal{C}_{n,b}$. We say that b is n-admissible if:

(a) $b \neq 0, \pm 1$;

(b) $\operatorname{char} \mathbb{F}_q \nmid m$;

(c) $X_{n,b}$ is a smooth absolutely irreducible plane curve of degree m;

(d) the five lines

$$W = 2Z, \qquad W = -2Z, \qquad V = 2Z, \qquad V = -2Z, \qquad Z = 0$$

meet $X_{n,b}$ transversely;

(e) the four strict-line intersection divisors corresponding to

$$W = 2Z, \qquad W = -2Z, \qquad V = 2Z, \qquad V = -2Z$$

are pairwise disjoint.

Lemma 5.3 (Strict divisor parity)**.** *Assume b is n-admissible. Then no nonempty product*

$$f_1^{e_1} f_2^{e_2} f_3^{e_3}, \qquad e_i \in \{0, 1\},$$

is a constant times a square in $\overline{\mathbb{F}}_q(X_{n,b})$.

Proof. Order the four strict divisors as

$$D_{W,+}, \quad D_{W,-}, \quad D_{V,+}, \quad D_{V,-},$$

corresponding respectively to

$$W = 2Z, \quad W = -2Z, \quad V = 2Z, \quad V = -2Z.$$

By admissibility, these divisors are reduced and pairwise disjoint.

The functions

$$f_1 = W^2 - 4, \qquad f_2 = V^2 - 4, \qquad f_3 = (W + 2)(V + 2)$$

have the following parity vectors along these four strict divisors:

$$f_1 : \quad (1, 1, 0, 0),$$

$$f_2 : \quad (0, 0, 1, 1),$$

and

$$f_3 : \quad (0, 1, 0, 1).$$

These three vectors are linearly independent over $\mathbb{F}_2$. Indeed, if

$$e_1(1, 1, 0, 0) + e_2(0, 0, 1, 1) + e_3(0, 1, 0, 1) = 0,$$

then the first coordinate gives $e_1 = 0$, the third coordinate gives $e_2 = 0$, and then the second coordinate gives $e_3 = 0$.

Therefore every nonempty product $f_1^{e_1} f_2^{e_2} f_3^{e_3}$ has odd valuation along at least one strict divisor. A square has even valuation at every divisor. Hence no such nonempty product is a constant times a square in $\overline{\mathbb{F}}_q(X_{n,b})$. □

6 General character-sum collision theorem

We now prove the main collision-counting theorem.

The proof of the collision lower bound has three ingredients. First, the desired strict lifting conditions are encoded by a quadratic-character indicator. Second, admissibility and the divisor-parity lemma ensure that all seven nontrivial character sums in the expansion are genuinely nontrivial. Third, the Weil bound for quadratic covers of the smooth projective trace curve controls these seven sums. The resulting positive lower bound produces an actual off-diagonal collision after applying the strict lifting lemma.

Theorem 6.1 (Admissible collision lower bound)**.** *Let $n = 2m + 1$, and assume b is n-admissible. Let $N_{n,b}$ denote the number of affine points $(W, V) \in \mathcal{C}_{n,b}(\mathbb{F}_q)$ satisfying*

$$\chi(W^2 - 4) = -1, \qquad \chi(V^2 - 4) = -1, \qquad \chi((W + 2)(V + 2)) = 1.$$

Then

$$N_{n,b} \geq \frac{q - 2(n-2)^2\sqrt{q} - (20n - 19)}{8}.$$

Proof. Let $X = X_{n,b}$. Since b is admissible, X is a smooth absolutely irreducible plane curve of degree m. Hence its genus is

$$g = \frac{(m-1)(m-2)}{2}.$$

On the affine curve, set

$$f_1 = W^2 - 4, \qquad f_2 = V^2 - 4, \qquad f_3 = (W + 2)(V + 2).$$

Away from the zeros of $f_1 f_2 f_3$, the indicator for the desired sign pattern is

$$I(P) = \frac{1}{8}(1 - \chi(f_1(P)))(1 - \chi(f_2(P)))(1 + \chi(f_3(P))).$$

Thus $I(P) = 1$ exactly when

$$\chi(f_1(P)) = -1, \qquad \chi(f_2(P)) = -1, \qquad \chi(f_3(P)) = 1,$$

and $I(P) = 0$ otherwise.

Expanding the indicator gives one main term and seven nontrivial character sums, corresponding to

$$f_1, \quad f_2, \quad f_3, \quad f_1 f_2, \quad f_1 f_3, \quad f_2 f_3, \quad f_1 f_2 f_3.$$

By Lemma 5.3, none of these functions is a constant times a square in $\overline{\mathbb{F}}_q(X)$. Therefore each associated quadratic character sum is nontrivial.

We use the standard quadratic character-sum estimate on a smooth projective curve: if $F \in \mathbb{F}_q(X)$ is not a constant times a square and has odd branch support of size r, then

$$\left| \sum_{P \in X(\mathbb{F}_q)} \chi(F(P)) \right| \leq (2g - 2 + r)\sqrt{q}.$$

Here r is the number of geometric places at which F has odd valuation. This is the standard conductor-degree form of the Weil bound for the quadratic cover $Y^2 = F$; see Rosen [8, Chapter 9] or Stichtenoth [9].

By admissibility, the line at infinity $Z = 0$ meets X transversely. Since each f_i has projective denominator Z^2, every pole of every f_i at infinity has order 2. Hence the points at infinity do not contribute to the odd branch support. Thus the odd branch support is controlled by the strict lines. The product $f_1 f_2$ has odd support on all four strict lines. Each strict line meets a degree-m curve in at most m geometric points, so this support has size at most $4m$. The other six nontrivial products have odd support on two strict lines and hence support size at most $2m$.

Also, Weil's bound gives

$$\#X(\mathbb{F}_q) \geq q + 1 - 2g\sqrt{q}.$$

Therefore the total square-root coefficient in the lower bound is

$$2g + 6(2g - 2 + 2m) + (2g - 2 + 4m).$$

Using

$$g = \frac{(m-1)(m-2)}{2},$$

we have

$$2g = m^2 - 3m + 2,$$

$$2g - 2 + 2m = m^2 - m,$$

and

$$2g - 2 + 4m = m^2 + m.$$

Thus

$$2g + 6(2g - 2 + 2m) + (2g - 2 + 4m) = 8m^2 - 8m + 2.$$

Since $n = 2m + 1$, this is

$$2(n-2)^2.$$

It remains to account for points where the affine strict indicator is not used. The four strict lines contribute at most $4m$ projective points, and the line at infinity contributes at most m projective points. Thus at most $5m$ points are removed from the main point count. Since the indicator has denominator 8, this contributes at most $40m$ to the numerator of the lower bound. Absorbing the harmless $+1$ from the projective point-count estimate gives the safe correction

$$40m + 1 = 20n - 19.$$

This correction is deliberately conservative; sharpness of this constant is not needed for the positivity result. Hence

$$N_{n,b} \geq \frac{q - 2(n-2)^2\sqrt{q} - (20n - 19)}{8}.$$

□

Corollary 6.2 (Positivity threshold)**.** *Assume b is n-admissible. If*

$$q > \left((n-2)^2 + \sqrt{(n-2)^4 + 20n - 19}\right)^2,$$

then $N_{n,b} > 0$. Consequently, for every nondegenerate a, the polynomial $F_{n,a,b}$ is not a permutation polynomial of $\mathbb{F}_{q^2}$.

Proof. Let $t = \sqrt{q}$. The lower bound in Theorem 6.1 is positive when

$$t^2 - 2(n-2)^2 t - (20n - 19) > 0.$$

The positive root is

$$t = (n-2)^2 + \sqrt{(n-2)^4 + 20n - 19}.$$

Therefore the stated hypothesis implies $N_{n,b} > 0$. By Lemma 5.1, a strict trace point lifts to an off-diagonal collision on μ_{q+1}. Hence the induced rational map is not injective on μ_{q+1}, so $F_{n,a,b}$ cannot permute $\mathbb{F}_{q^2}$. □

7 Effective exceptional locus and general sparsity

We now make admissibility effective. Let

$$F_{m,b}(W,V) = bD_m(W) + S_m(V) - b^2 S_{m-1}(V),$$

and let

$$F^h_{m,b}(W,V,Z)$$

be its projective homogenization.

We impose the standing condition

$$\operatorname{char}\mathbb{F}_q \nmid m$$

in this section. This avoids inseparable degeneration of the leading Dickson term.

Remark 7.1. The hypothesis $\operatorname{char}\mathbb{F}_q \nmid m$ is used only in the effective exceptional-locus construction. It prevents inseparable degeneration of the degree-m Dickson leading term. The universal collision identity and the formal Dickson trace reduction are polynomial identities and remain meaningful without this hypothesis. What may fail in bad characteristic is the smooth plane-curve discriminant criterion used to certify admissibility uniformly in b.

Let $\Delta_{m,q}(b) \in \mathbb{F}_q[b]$ denote the projective discriminant of $F^h_{m,b}$, formed algebraically and then reduced in $\mathbb{F}_q[b]$. Let

$$T^{(m)}_{W,+}(b), \quad T^{(m)}_{W,-}(b), \quad T^{(m)}_{V,+}(b), \quad T^{(m)}_{V,-}(b)$$

denote the discriminants of the binary forms obtained by restricting $F^h_{m,b}$ to the strict lines

$$W = 2Z, \quad W = -2Z, \quad V = 2Z, \quad V = -2Z.$$

Finally, define the corner product

$$C_m(b) = \prod_{\epsilon,\delta\in\{\pm1\}} F_{m,b}(2\epsilon, 2\delta).$$

Definition 7.2 (Effective exceptional polynomial)**.** Define

$$E_{n,q}(b) = b(b^2-1)\Delta_{m,q}(b)T^{(m)}_{W,+}(b)T^{(m)}_{W,-}(b)T^{(m)}_{V,+}(b)T^{(m)}_{V,-}(b)C_m(b).$$

We view $E_{n,q}(b)$ as an explicit sufficient exceptional polynomial. We do not claim that it is minimal.

Remark 7.3. The polynomial $E_{n,q}(b)$ should be viewed as an effective sufficient exceptional polynomial. Its vanishing records every way in which the simple admissibility argument used in this paper may fail: universal degeneracy, projective singularity, tangency to one of the strict lines, or intersection with a strict corner. Some roots of $E_{n,q}$ may therefore be artifacts of the method rather than parameters producing genuine permutation polynomials. This is why the sparsity theorem bounds possible permutation parameters, not actual permutation parameters.

Proposition 7.4 (Nonvanishing implies admissibility)**.** *Assume* $\operatorname{char}\mathbb{F}_q \nmid m$*. If*

$$E_{n,q}(b) \neq 0,$$

then b is n-admissible.

Proof. The factor $b(b^2-1)$ gives

$$b \neq 0, \qquad b \neq \pm 1.$$

The assumed characteristic condition gives

$$\operatorname{char} \mathbb{F}_q \nmid m.$$

The nonvanishing of the projective discriminant $\Delta_{m,q}(b)$ implies that the projective plane curve $X_{n,b}$ is smooth. A smooth projective plane curve is absolutely irreducible, since a reducible plane curve over $\overline{\mathbb{F}}_q$ would have component intersections and hence singularities.

Since $\operatorname{char} \mathbb{F}_q \nmid m$, the leading form

$$bW^m + V^m$$

is separable on the line at infinity $Z = 0$. Hence the line at infinity meets $X_{n,b}$ transversely. Moreover, the points

$$[0 : 1 : 0], \qquad [1 : 0 : 0]$$

do not lie on $X_{n,b}$, since the leading form evaluates to 1 and b, respectively.

The nonvanishing of the four strict-line discriminants implies that the restrictions of $F^h_{m,b}$ to the strict lines have no repeated roots. Therefore the four strict lines meet $X_{n,b}$ transversely.

The nonvanishing of $C_m(b)$ shows that $X_{n,b}$ avoids the four affine corner points where one W-strict line and one V-strict line meet. The two pairs of parallel strict lines meet at the two points at infinity displayed above, and these points are not on $X_{n,b}$. Hence the strict-line divisors are pairwise disjoint.

Thus all admissibility conditions hold. □

Proposition 7.5 (Degree bound for the exceptional polynomial)**.** *The exceptional polynomial satisfies the conservative degree bound*

$$\deg E_{n,q} \leq \frac{3n^2 - 2n + 1}{2}.$$

Equivalently, for $n = 2m + 1$,

$$\deg E_{n,q} \leq 6m^2 + 4m + 1.$$

Proof. The universal factor $b(b^2-1)$ has degree 3.

The projective discriminant of a plane curve of degree m is homogeneous of degree $3(m-1)^2$ in the coefficients; see Gelfand–Kapranov–Zelevinsky [10]. The coefficients of $F^h_{m,b}$ have degree at most 2 in b. Hence the contribution of the projective discriminant to the b-degree is at most

$$6(m-1)^2.$$

Each strict-line restriction is a binary form of degree m whose coefficients have b-degree at most 2. The discriminant of a degree m binary form has degree $2m - 2$ in the coefficients, so each strict-line discriminant has b-degree at most

$$4m - 4.$$

There are four such discriminants, contributing at most

$$16m - 16.$$

The corner product has four factors, each of b-degree at most 2, hence degree at most 8. Adding all contributions gives

$$3 + 6(m-1)^2 + 16m - 16 + 8 = 6m^2 + 4m + 1.$$

Since $n = 2m + 1$,

$$6m^2 + 4m + 1 = \frac{3n^2 - 2n + 1}{2}.$$

□

The bound in Proposition 7.5 is intentionally conservative. It uses general degree bounds for dense ternary and binary discriminants, while the trace polynomials $F_{m,b}$ are highly sparse and depend on b through very few coefficient functions. The explicit computations for $n = 7$ and $n = 9$ show that the actual nondegenerate exceptional degree can be far smaller than the general bound. The purpose of the general estimate is not sharpness; it is to prove uniform $O_n(1/q)$ sparsity for each fixed odd degree.

Theorem 7.6 (Asymptotic sparsity of possible permutation fibers)**.** *Fix an odd integer $n = 2m+1$. Assume*

$$\operatorname{char} \mathbb{F}_q \nmid m$$

and that $E_{n,q}(b)$ is not the zero polynomial in $\mathbb{F}_q[b]$. For q satisfying the positivity threshold in Corollary 6.2, possible nondegenerate permutation pairs (a, b) are contained in at most

$$\frac{3n^2 - 2n + 1}{2}$$

vertical b-fibers. Consequently, their density in the nondegenerate parameter space is at most

$$\frac{3n^2 - 2n + 1}{2(q-3)}.$$

In particular, for fixed n, this density is $O_n(1/q)$.

Proof. By Proposition 7.4, if $E_{n,q}(b) \neq 0$, then b is admissible. By Corollary 6.2, such a b forces an off-diagonal collision on μ_{q+1}, so no nondegenerate a can yield a permutation polynomial. Hence possible permutation examples must lie over roots of $E_{n,q}(b)$.

Since $E_{n,q}$ is not the zero polynomial, it has at most

$$\deg E_{n,q} \leq \frac{3n^2 - 2n + 1}{2}$$

roots in $\mathbb{F}_q$. For each fixed nondegenerate b, there are at most $q - 2$ choices of a satisfying $a \neq \pm b$. Thus the number of possible pairs is at most

$$\frac{3n^2 - 2n + 1}{2}(q-2).$$

The full nondegenerate parameter space has size

$$(q-3)(q-2),$$

because $b \notin \{0, \pm 1\}$ and $a \neq \pm b$. Dividing gives the stated density bound. □

8 The first odd-degree trace curves

The first few odd degrees make the Dickson trace framework concrete.

Proposition 8.1 (First odd-degree trace curves)**.** *The first odd-degree Dickson trace curves are as follows:*

$$n = 3: \quad bW + V + 1 - b^2 = 0.$$

$$n = 5: \quad bW^2 + V^2 + (1 - b^2)V - (b + 1)^2 = 0.$$

$$n = 7: \quad b(W^3 - 3W) + V^3 + (1 - b^2)V^2 - (b^2 + 2)V + (b^2 - 1) = 0.$$

$$n = 9: \quad bW^4 - 4bW^2 + V^4 + (1 - b^2)V^3 - (b^2 + 3)V^2 + (2b^2 - 2)V + b^2 + 2b + 1 = 0.$$

Proof. The formulas follow by substituting $m = 1, 2, 3, 4$ into

$$bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

The relevant Dickson polynomials are

$$D_1(X) = X,$$

$$D_2(X) = X^2 - 2,$$

$$D_3(X) = X^3 - 3X,$$

and

$$D_4(X) = X^4 - 4X^2 + 2.$$

Also

$$S_1(X) = 1 + X,$$

$$S_2(X) = X^2 + X - 1,$$

$$S_3(X) = X^3 + X^2 - 2X - 1,$$

and

$$S_4(X) = X^4 + X^3 - 3X^2 - 2X + 1.$$

Substitution gives the displayed equations. □

Degree n	$m = (n - 1)/2$	Trace curve	Role
3	1	line	elementary base case
5	2	conic	complete classification in the companion paper
7	3	cubic	first explicit higher-degree case
9	4	quartic	second explicit higher-degree case

Table 1: The first odd-degree Dickson trace curves.

The $n = 5$ conic is the curve underlying the complete degree-five classification in [11]. Beginning with $n = 7$, the trace curves have positive genus in the smooth projective case, and full classification of all exceptional fibers becomes substantially more difficult. The next two sections work out explicit generic non-permutation theorems for the cubic and quartic cases.

9 The degree-seven cubic

For $n = 7$, we have $m = 3$. The trace curve is the cubic

$$\mathcal{E}_b : \quad b(W^3 - 3W) + V^3 + (1 - b^2)V^2 - (b^2 + 2)V + (b^2 - 1) = 0.$$

Let

$$Q_+(b) = 5b^4 + 8b^3 + 18b^2 + 28b + 49,$$

and

$$Q_-(b) = 5b^4 - 8b^3 + 18b^2 - 28b + 49.$$

Define

$$E_7(b) = b(b^2 - 1)(25b^2 - 49)Q_+(b)Q_-(b).$$

Proposition 9.1 (Degree-seven admissibility)**.** *Assume*

$$\operatorname{char} \mathbb{F}_q \neq 3.$$

If

$$E_7(b) \neq 0,$$

then the degree-seven cubic $\mathcal{E}_b$ is 7-admissible.

Proof. Let

$$F_{7,b}(W, V) = b(W^3 - 3W) + V^3 + (1 - b^2)V^2 - (b^2 + 2)V + (b^2 - 1).$$

Since $E_7(b) \neq 0$, we have $b \neq 0, \pm 1$.

We first check smoothness. The affine W-partial derivative is

$$\frac{\partial F_{7,b}}{\partial W} = 3b(W^2 - 1).$$

Since $\operatorname{char} \mathbb{F}_q \neq 3$ and $b \neq 0$, any affine singularity must have $W = \pm 1$. Eliminating V between $F_{7,b}(1, V)$ and $\partial F_{7,b}(1, V)/\partial V$ gives

$$\operatorname{Res}_V(F_{7,b}(1, V), F_{7,b,V}(1, V)) = -(b - 1)^2(b + 1)^2 Q_-(b).$$

Similarly,

$$\operatorname{Res}_V(F_{7,b}(-1, V), F_{7,b,V}(-1, V)) = -(b - 1)^2(b + 1)^2 Q_+(b).$$

Both are nonzero when $E_7(b) \neq 0$, so there are no affine singularities.

At infinity, the leading form is

$$bW^3 + V^3.$$

Its W- and V-partials are

$$3bW^2 \qquad \text{and} \qquad 3V^2.$$

Since $\operatorname{char} \mathbb{F}_q \neq 3$ and $b \neq 0$, a singular point at infinity would force $W = V = 0$, impossible in projective space. Hence the projective cubic is smooth, and therefore absolutely irreducible. Moreover, the leading form $bW^3 + V^3$ is separable on $Z = 0$, so the line at infinity meets the cubic transversely.

Next we check strict-line transversality. The strict-line discriminants are

$$W = 2: \quad (b-1)^2(b+1)^2 Q_+(b),$$

$$W = -2: \quad (b-1)^2(b+1)^2 Q_-(b),$$

$$V = 2: \quad -27b^2(b-1)(b+1)(5b-7)(5b+7),$$

and

$$V = -2: \quad -27b^2(b-1)^2(b+1)^2.$$

All are nonzero when $E_7(b) \neq 0$. Therefore the four strict lines meet the cubic transversely.

Finally, the corner values are

$$F_{7,b}(2,2) = -(b+1)(5b-7),$$

$$F_{7,b}(2,-2) = -(b-1)^2,$$

$$F_{7,b}(-2,2) = -(b-1)(5b+7),$$

and

$$F_{7,b}(-2,-2) = -(b+1)^2.$$

These are nonzero under the same hypothesis. Thus the curve avoids the four affine strict corners. The two pairs of parallel strict lines meet at infinity, and the leading form $bW^3 + V^3$ does not vanish at the two corresponding points when $b \neq 0$. Hence the strict divisors are pairwise disjoint.

All admissibility conditions hold. □

Theorem 9.2 (Generic degree-seven non-permutation theorem)**.** *Let q be an odd prime power, and let $b \in \mathbb{F}_q$ satisfy*

$$b \neq 0, \qquad b \neq \pm 1.$$

Assume

$$\operatorname{char} \mathbb{F}_q \neq 3, \qquad E_7(b) \neq 0, \qquad q > 3000.$$

Then for every $a \in \mathbb{F}_q$ with $a \neq \pm b$, the polynomial

$$F_{7,a,b}(x) = x^7 + ax^{q+6} + bx^{6q+1} + \frac{a}{b}x^{7q}$$

is not a permutation polynomial of $\mathbb{F}_{q^2}$.

Proof. For $n = 7$, Theorem 6.1 gives

$$N_{7,b} \geq \frac{q - 50\sqrt{q} - 121}{8}.$$

The right-hand side is positive for $q > 3000$. By Proposition 9.1, the hypothesis $E_7(b) \neq 0$ implies admissibility. Hence $N_{7,b} > 0$, so Lemma 5.1 produces an off-diagonal collision on μ_{q+1}. Therefore $F_{7,a,b}$ cannot permute $\mathbb{F}_{q^2}$ for any $a \neq \pm b$. □

Corollary 9.3 (Degree-seven density bound)**.** *For $q > 3000$ and $\operatorname{char} \mathbb{F}_q \neq 3$, possible nondegenerate degree-seven permutation pairs (a, b) lie over at most 10 b-fibers. Their density in the nondegenerate degree-seven parameter space is at most*

$$\frac{10}{q-3}.$$

Proof. Inside the nondegenerate parameter space, the universal factors $b(b^2-1)$ are excluded. Thus the relevant exceptional factor is

$$(25b^2 - 49)Q_+(b)Q_-(b),$$

which has degree

$$2 + 4 + 4 = 10.$$

Hence there are at most 10 relevant exceptional b-values. For each such b, there are at most $q-2$ choices of a satisfying $a \neq \pm b$. Thus possible permutation pairs are at most $10(q-2)$. The full nondegenerate parameter space has size $(q-3)(q-2)$. Dividing gives the stated density bound. □

10 The degree-nine quartic

For $n = 9$, we have $m = 4$. The trace curve is the quartic

$$\mathcal{C}_{9,b}: \quad bW^4 - 4bW^2 + V^4 + (1-b^2)V^3 - (b^2+3)V^2 + (2b^2-2)V + b^2 + 2b + 1 = 0.$$

Let

$$P_+(b) = 49b^6 + 70b^5 + 151b^4 + 212b^3 + 351b^2 + 486b + 729,$$

and

$$P_-(b) = 49b^6 - 70b^5 + 151b^4 - 212b^3 + 351b^2 - 486b + 729.$$

Define

$$E_9(b) = b(b^2-1)(7b-9)(7b+9)P_+(b)P_-(b).$$

Proposition 10.1 (Degree-nine admissibility)**.** *If*

$$E_9(b) \neq 0,$$

then the degree-nine quartic $\mathcal{C}_{9,b}$ is 9-admissible.

Proof. Let

$$\begin{aligned} F_{9,b}(W,V) =& bW^4 - 4bW^2 + V^4 + (1-b^2)V^3 \\ & - (b^2+3)V^2 + (2b^2-2)V + b^2 + 2b + 1. \end{aligned}$$

Since $E_9(b) \neq 0$, we have $b \neq 0, \pm 1$. Since q is odd, $\operatorname{char} \mathbb{F}_q \nmid 4$.

We first check smoothness. The affine W-partial derivative is

$$\frac{\partial F_{9,b}}{\partial W} = 4bW(W^2-2).$$

Since q is odd and $b \neq 0$, an affine singularity must lie on one of the branches

$$W = 0 \qquad \text{or} \qquad W^2 = 2.$$

For $W = 0$, eliminating V between $F_{9,b}(0,V)$ and $\partial F_{9,b}(0,V)/\partial V$ gives

$$(b-1)^4(b+1)^2P_+(b).$$

For the branch $W^2 = 2$, substitute $W^2 = 2$ into $F_{9,b}(W, V)$ and into $\partial F_{9,b}(W, V)/\partial V$, and then eliminate V. The resulting branch resultant is

$$(b-1)^2(b+1)^4 P_-(b).$$

Both are nonzero when $E_9(b) \neq 0$. Hence there are no affine singularities.

At infinity, the leading form is

$$bW^4 + V^4.$$

Its W- and V-partials are

$$4bW^3 \qquad \text{and} \qquad 4V^3.$$

Since q is odd and $b \neq 0$, a singular point at infinity would force $W = V = 0$, impossible in projective space. Thus the projective quartic is smooth, and hence absolutely irreducible. Moreover, the leading form $bW^4 + V^4$ is separable on $Z = 0$, so the line at infinity meets the quartic transversely.

Next we check strict-line transversality. The strict-line discriminants are

$$\operatorname{disc}_V F_{9,b}(2, V) = \operatorname{disc}_V F_{9,b}(-2, V) = (b-1)^4(b+1)^2 P_+(b),$$

$$\operatorname{disc}_W F_{9,b}(W, 2) = -256b^3(b-1)^2(b+1)(7b-9)(7b+9)^2,$$

and

$$\operatorname{disc}_W F_{9,b}(W, -2) = 256b^3(b-1)^4(b+1)^2.$$

All are nonzero when $E_9(b) \neq 0$. Therefore the four strict lines meet the quartic transversely.

Finally, the corner values are

$$F_{9,b}(2, 2) = F_{9,b}(-2, 2) = -(b+1)(7b-9),$$

and

$$F_{9,b}(2, -2) = F_{9,b}(-2, -2) = (b+1)^2.$$

These are nonzero under the same hypothesis. Hence the curve avoids all four affine strict corners. At infinity, the two pairs of parallel strict lines meet at

$$[0 : 1 : 0] \qquad \text{and} \qquad [1 : 0 : 0],$$

and the leading form $bW^4 + V^4$ does not vanish at either point when $b \neq 0$. Thus the strict divisors are pairwise disjoint.

All admissibility conditions hold. □

Theorem 10.2 (Generic degree-nine non-permutation theorem)**.** *Let q be an odd prime power, and let $b \in \mathbb{F}_q$ satisfy*

$$b \neq 0, \qquad b \neq \pm 1.$$

Assume

$$E_9(b) \neq 0, \qquad q > 10000.$$

Then for every $a \in \mathbb{F}_q$ with $a \neq \pm b$, the polynomial

$$F_{9,a,b}(x) = x^9 + ax^{q+8} + bx^{8q+1} + \frac{a}{b}x^{9q}$$

is not a permutation polynomial of $\mathbb{F}_{q^2}$.

Proof. For $n = 9$, Theorem 6.1 gives

$$N_{9,b} \geq \frac{q - 98\sqrt{q} - 161}{8}.$$

The right-hand side is positive for

$$q > \left(49 + \sqrt{2562}\right)^2,$$

and in particular for $q > 10000$. By Proposition 10.1, the condition $E_9(b) \neq 0$ implies admissibility. Hence $N_{9,b} > 0$, so Lemma 5.1 produces an off-diagonal collision on μ_{q+1}. Therefore $F_{9,a,b}$ cannot permute $\mathbb{F}_{q^2}$ for any $a \neq \pm b$. □

Corollary 10.3 (Degree-nine density bound)**.** *For $q > 10000$, possible nondegenerate degree-nine permutation pairs (a, b) lie over at most 14 b-fibers. Their density in the nondegenerate degree-nine parameter space is at most*

$$\frac{14}{q-3}.$$

Proof. Inside the nondegenerate parameter space, the universal factors $b(b^2 - 1)$ are excluded. Thus the relevant exceptional factor is

$$(7b - 9)(7b + 9)P_+(b)P_-(b),$$

which has degree

$$1 + 1 + 6 + 6 = 14.$$

Hence there are at most 14 relevant exceptional b-values. For each such b, there are at most $q - 2$ choices of a satisfying $a \neq \pm b$. Thus possible permutation pairs are at most $14(q - 2)$. The full nondegenerate parameter space has size $(q - 3)(q - 2)$. Dividing gives the stated density bound. □

11 Explicit density bounds and a possible linear pattern

The explicit degree-seven and degree-nine computations show that the general degree bound in Proposition 7.5 is conservative.

For $n = 7$, the general bound gives

$$\frac{3(7)^2 - 2(7) + 1}{2} = 67.$$

However, the explicit nondegenerate exceptional factor has degree 10, giving density at most $10/(q - 3)$ in the large-field range of Theorem 9.2.

For $n = 9$, the general bound gives

$$\frac{3(9)^2 - 2(9) + 1}{2} = 113.$$

However, the explicit nondegenerate exceptional factor has degree 14, giving density at most $14/(q - 3)$ in the large-field range of Theorem 10.2.

After removing the universal degenerate factor $b(b^2 - 1)$, the first two higher-degree cases have nondegenerate exceptional degrees

$$10 = 2(7) - 4$$

Degree n	Trace curve	General degree bound	Explicit nondegenerate degree	Density bound
7	cubic	67	10	$10/(q-3)$
9	quartic	113	14	$14/(q-3)$

Table 2: General exceptional-degree bounds versus explicit nondegenerate exceptional degrees in the first two higher-degree cases.

and

$$14 = 2(9) - 4.$$

This suggests a possible linear exceptional-degree pattern.

Question 11.1. *For odd $n \geq 7$, does the effective nondegenerate exceptional locus often have degree $2n-4$, or do new higher-degree components appear for larger n?*

The next natural test case is $n = 11$. Then $m = 5$, and the trace curve is the quintic

$$\begin{aligned} \mathcal{C}_{11,b}: \quad & b(W^5 - 5W^3 + 5W) + V^5 + (1-b^2)V^4 - (b^2+4)V^3 \\ & + (3b^2-3)V^2 + (2b^2+3)V + (1-b^2) = 0. \end{aligned}$$

The displayed quintic provides the next natural computational test of the framework. The values 10 and 14 obtained for $n = 7$ and $n = 9$ suggest, but do not prove, that the nondegenerate exceptional degree may often be $2n - 4$. For $n = 11$, this would predict degree

$$2(11) - 4 = 18.$$

Whether this linear pattern persists is left as a future problem.

A second natural question concerns the exceptional fibers themselves.

Question 11.2. *When $E_n(b) = 0$, do the exceptional fibers contain genuine permutation polynomial families, or do they primarily record singularity, tangency, and boundary artifacts?*

The companion degree-five classification shows that exceptional fibers can contain true infinite permutation families. Understanding the corresponding exceptional fibers in higher odd degrees is a natural continuation of the present work.

12 Relation to previous work

Permutation polynomials of the form

$$x^r h(x^{q-1})$$

over $\mathbb{F}_{q^2}$ have been studied extensively through root-of-unity reductions and AGW-type criteria [1]. These reductions convert a full-field permutation problem into a coprimality condition together with a rational-map permutation problem on a subgroup of roots of unity. This approach underlies many modern constructions and classifications of sparse permutation polynomials over quadratic extensions; see, for example, [6, 2].

Much of the existing literature on sparse permutation polynomials focuses on constructing explicit families or classifying particular low-term supports. Quadrinomials over finite fields have been studied from several perspectives, including support patterns, coefficient constraints, and induced low-degree rational functions on μ_{q+1}; see [3, 4, 5]. The reciprocal quadrinomial family considered here is different in that the coefficient relation forces a universal collision factorization. Outside the natural degenerate cases $b = 0, \pm 1$ and $a = \pm b$, the off-diagonal collision equation becomes independent of a. This reduces the obstruction to permutation behavior to a one-parameter problem in b.

The degree-five member of this reciprocal family is treated completely in the companion classification paper [11]. In that case, the Dickson trace curve is a conic, and the finite-field character conditions can be resolved completely, producing both infinite families and sporadic cases. The present paper identifies the general odd-degree mechanism behind that conic case. For odd $n = 2m + 1$, the collision equation becomes a Dickson trace curve

$$\mathcal{C}_{n,b}: \quad bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

Thus the degree-five conic is the first nontrivial instance of a broader line-conic-cubic-quartic hierarchy.

The present work differs from complete classification papers in its objective. For higher odd degrees, the trace curves have increasing degree and genus, and full classification of all exceptional fibers becomes substantially harder. Instead, this paper proves that outside an explicit exceptional b-locus, admissible Dickson trace curves force strict lifted collisions for all sufficiently large fields. Consequently, possible nondegenerate permutation examples are asymptotically sparse in the full two-parameter family. The explicit degree-seven and degree-nine computations show that this framework is effective beyond the conic case, producing concrete cubic and quartic exceptional loci and sharp large-field density bounds.

13 Conclusion

We have shown that the reciprocal quadrinomial family

$$F_{n,a,b}(x) = x^n + ax^{q+n-1} + bx^{(n-1)q+1} + \frac{a}{b}x^{nq}$$

has a universal collision structure. Outside the natural degenerate cases

$$b = 0, \qquad b = \pm 1, \qquad a = \pm b,$$

the off-diagonal collision equation is independent of the coefficient a. This is the algebraic feature that makes a one-parameter trace-curve analysis possible.

For odd degrees $n = 2m + 1$, the collision equation becomes the Dickson trace curve

$$\mathcal{C}_{n,b}: \quad bD_m(W) + S_m(V) - b^2 S_{m-1}(V) = 0.$$

This converts the unit-circle permutation problem into a geometric and character-sum problem on algebraic curves over $\mathbb{F}_q$. For admissible parameters b, the character-sum argument forces strict lifted collisions for all sufficiently large fields. As a result, possible nondegenerate permutation examples must lie over an explicit exceptional b-locus.

The general theory gives asymptotic sparsity for every fixed odd degree. The explicit degree-seven and degree-nine cases show that the method is effective beyond the degree-five conic case. For $n = 7$, the trace curve is a cubic and the nondegenerate exceptional degree is 10. For $n = 9$, the trace curve is a quartic and the nondegenerate exceptional degree is 14. These give large-field density bounds $10/(q-3)$ and $14/(q-3)$, respectively.

The explicit degrees 10 and 14 match $2n-4$ for $n = 7$ and $n = 9$. It remains to determine whether this possible linear pattern persists in higher odd degrees or whether new exceptional components appear. The next natural test case is $n = 11$, where the Dickson trace curve is a quintic.

More broadly, the reciprocal quadrinomial family appears to have two layers. The first is a universal geometric layer, where admissible trace curves force collisions and eliminate permutation behavior. The second is a thinner arithmetic exceptional layer, where genuinely rare permutation examples may live. Understanding those exceptional fibers is the natural continuation of the complete degree-five classification and the asymptotic sparsity framework developed here.

A Symbolic verification data

Several identities in the paper are polynomial identities over $\mathbb{Z}[a, b, x, y]$ or $\mathbb{Z}[b, W, V]$, after clearing denominators where necessary. They can therefore be checked symbolically before reduction modulo finite fields. The finite-field hypotheses in the main theorems, such as odd characteristic, $\operatorname{char}\mathbb{F}_q \nmid m$, and nonvanishing of the displayed exceptional factors, are stated separately in the corresponding results.

The symbolic verification files accompanying this manuscript are:

`verify_dickson_trace_framework.py`,

`symbolic_output.txt`,

and

`README.md`.

The verification is symbolic, not an exhaustive finite-field search. The symbolic verification does not replace the geometric and character-sum arguments in the proof. Its role is to confirm the polynomial identities, resultants, discriminants, and corner factorizations used in the explicit $n = 7$ and $n = 9$ cases. The Weil-bound arguments and admissibility implications are proved in the text.

The script checks the universal collision identity

$$h^*_{n,a,b}(x)h_{n,a,b}(y) - h^*_{n,a,b}(y)h_{n,a,b}(x) = -\frac{(a-b)(a+b)}{b^2}(x-y)K_{n,b}(x, y)$$

for

$$n = 3, 5, 7, 9.$$

It also checks the explicit Dickson trace equations for $n = 7$ and $n = 9$.

For $n = 7$, the script verifies the displayed singularity resultants, strict-line discriminants, and corner values leading to

$$E_7(b) = b(b^2-1)(25b^2-49)Q_+(b)Q_-(b).$$

For $n = 9$, the script verifies the corner values

$$F_{9,b}(2, 2) = F_{9,b}(-2, 2) = -(b+1)(7b-9),$$

$$F_{9,b}(2,-2) = F_{9,b}(-2,-2) = (b+1)^2,$$

the strict-line discriminants

$$\operatorname{disc}_V F_{9,b}(2,V) = \operatorname{disc}_V F_{9,b}(-2,V) = (b-1)^4(b+1)^2 P_+(b),$$

$$\operatorname{disc}_W F_{9,b}(W,2) = -256b^3(b-1)^2(b+1)(7b-9)(7b+9)^2,$$

and

$$\operatorname{disc}_W F_{9,b}(W,-2) = 256b^3(b-1)^4(b+1)^2.$$

It also verifies the affine smoothness resultants

$$\operatorname{Res}_V(F_{9,b}(0,V), F_{9,b,V}(0,V)) = (b-1)^4(b+1)^2 P_+(b),$$

and, after restricting to the branch $W^2 = 2$,

$$\operatorname{Res}_V\left(F_{9,b}(W,V)\big|_{W^2=2}, F_{9,b,V}(W,V)\big|_{W^2=2}\right) = (b-1)^2(b+1)^4 P_-(b),$$

together with the displayed factorization

$$E_9(b) = b(b^2-1)(7b-9)(7b+9)P_+(b)P_-(b).$$

The expected output terminates with

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ALL SYMBOLIC CHECKS PASSED.
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References

[1] A. Akbary, D. Ghioca, and Q. Wang, *On constructing permutations of finite fields*, Finite Fields and Their Applications **17** (2011), 51–67.

[2] K. Li, L. Qu, and Q. Wang, *New constructions of permutation polynomials of the form $x^r h(x^{q-1})$ over $\mathbb{F}_{q^2}$*, Designs, Codes and Cryptography **86** (2018), 2379–2405.

[3] Z. Ding and M. E. Zieve, *Determination of a class of permutation quadrinomials*, Proceedings of the London Mathematical Society **126** (2023), 874–918.

[4] F. Özbudak and B. Gülmez Temür, *Classification of some quadrinomials over finite fields of odd characteristic*, Finite Fields and Their Applications **87** (2023), 102158.

[5] K. Garg, S. U. Hasan, C. Li, H. Kumar, and M. Pal, *Permutation polynomials with a few terms over finite fields*, Advances in Mathematics of Communications **21** (2026), 15–41.

[6] X. Hou, *Permutation polynomials over finite fields—a survey of recent advances*, Finite Fields and Their Applications **32** (2015), 82–119.

[7] R. Lidl and H. Niederreiter, *Finite Fields*, Encyclopedia of Mathematics and its Applications, Vol. 20, Cambridge University Press, Cambridge, 1997.

[8] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics, Vol. 210, Springer, New York, 2002.

[9] H. Stichtenoth, *Algebraic Function Fields and Codes*, Graduate Texts in Mathematics, Vol. 254, Springer, Berlin, 2nd ed., 2009.

[10] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[11] B. M. Woody, *A Complete Classification of a Reciprocal Degree-Five Quadrinomial Family over $\mathbb{F}_{q^2}$*, preprint, 2026.