1.3 Absolute Value Equations and Inequalities

The absolute value is a simple concept that is often misunderstood. All you need to remember is that the absolute value takes all values and makes them positive. That is, $|-1| = 1$ and $|1| = 1$. When solving a linear absolute value equation or inequality there will always be two solutions. In fact, the mantra you want to repeat to yourself in this section is "the absolute value has two solutions." For a linear absolute value equation we have that $|mx + b| = c$ is identical to $mx + b = \pm c$. In this section we must remind ourselves of the value of effective practice. It is essential that we practice these problems verbatim as to how they are laid out here. This will also help you to avoid long-term confusion. Remember that perfect practice makes perfect and quality repetition is the key to success.

Solving Absolute Value Equations.

Example 1: Solve the given equation for x.

$|2x + 7| = 3$

We separate the absolute value into two columns: one positive and one negative.

$2x + 7 = 2$	**or**	$-(2x + 7) = 2$
$2x = 2 - 7$		$2x + 7 = -2$
$2x = -5$		$2x = -2 - 7$
$x = -\frac{5}{2}$		$2x = -9$
		$x = -\frac{9}{2}$

The two solutions are $x = -\frac{5}{2}$ and $x = -\frac{9}{2}$.

Example 2: Solve the given equation for x.

$|3x + 4| + 2 = 17$

Whenever possible, you want to have an absolute value on one side of the equation and all other terms combined and reduced on the other side.

$|3x + 4| = 17 - 2$

$|3x + 4| = 15$

$3x + 4 = 15$	**or**	$-(3x + 4) = 15$
$3x = 15 - 4$		$3x + 4 = -15$
$3x = 11$		$3x = -15 - 4$
$x = \frac{11}{3}$		$3x = -19$
		$x = -\frac{19}{3}$

The two solutions are $x = -\frac{11}{3}$ **and** $x = -\frac{19}{3}$**.**

The first step in the second column may seem unnecessary, but it will prove to be very necessary when we encounter absolute value inequality problems. These minute details are what we need to focus on in order to succeed in calculus.

Let us see what happens when there is an absolute value on both sides of the equation. Here the trick is not to overthink the problem. Practice the mechanics first. Think about the problem later, when it is mastered.

Example 3: Solve the given equation for x**.**

$|x - 4| = |2 - 4x|$

We proceed exactly as we did in the previous two examples.

$x - 4 = 2 - 4x$	**or**	$-(x - 4) = 2 - 4x$
$x + 4x = 2 + 4$		$-x + 4 = 2 - 4x$
$5x = 6$		$-x + 4x = 2 - 4$
$x = \frac{6}{5}$		$3x = -2$
		$x = -\frac{2}{3}$

The two solutions are $x = \frac{6}{5}$ and $x = -\frac{2}{3}$.

One can always substitute the solutions back into the original equation to check if it is correct.

Suppose that $x = \frac{6}{5}$.

$|\frac{6}{5} - 4| = |2 - 4 \cdot \frac{6}{5}|$

$|\frac{6}{5} - \frac{20}{5}| = |\frac{10}{5} - \frac{24}{5}|$

$| - \frac{14}{5} = | - \frac{14}{5}|$

$\frac{14}{5} = \frac{14}{5}$

Therefore the solution $x = \frac{6}{5}$ checks out .

Now suppose that $x = -\frac{2}{3}$.

$| - \frac{2}{3} - 4| = |2 - 4 \cdot \frac{-2}{3}|$

$| - \frac{2}{3} - \frac{12}{3}| = |\frac{6}{3} + \frac{8}{3}|$

$| - \frac{14}{3}| = |\frac{14}{3}|$

$\frac{14}{3} = \frac{14}{3}$

Therefore the solution $x = -\frac{2}{3}$ checks out.

Solving Absolute Value Inequalities.

We continue to use the same method practiced in the previous examples here. It is important when practicing to not skip steps. When we skip steps in mathematics, we are no longer practicing mathematics.

Example 4: Solve the given inequality for x.

$|x| < 5$

$x < 5$	**or**	$-x < 5$
		$x > -5$

Notice that in the second row we multiplied by -1, so we flipped the inequality. This is important and exactly why we trained the way we did in the first three examples.

The above solution can be expressed as follows.

Set Notation: $\{\, x \mid -5 < x < 5 \,\}$ **.**

Interval Notation: $(-5,\, 5)$ **.**

Graphically on a Number Line:

Theorem: Suppose that a is any positive number.

$$|x| < a \;\Leftrightarrow\; -a < x < a$$

$$|x| \leq a \;\Leftrightarrow\; -a \leq x \leq a$$

$$|x| > a \;\Leftrightarrow\; x > a \text{ or } x < -a$$

$$|x| \geq a \;\Leftrightarrow\; x \geq a \text{ or } x \leq -a$$

Example 5: Solve the given inequality for x.

$|2x-3| \leq 9$

$2x-3 \leq 9$	**or**	$-(2x-3) \leq 9$
$2x \leq 9+3$		$2x-3 \geq -9$
$2x \leq 12$		$2x \geq -9+3$
$x \leq 6$		$2x \geq -6$
		$x \geq -3$

The above solution can be expressed as follows.

Set Notation: $\{\, x \mid -3 \leq x \leq 6 \,\}$ **.**

Interval Notation: $[-3,\ 6]$ **.**

Graphically on a Number Line:

Example 6: Solve the given inequality for x.

$|3x-8| > 16$

$3x-8 > 16$	**or**	$-(3x-8) > 16$
$3x > 16+8$		$3x-8 < -16$
$3x > 24$		$3x < -16+8$
$x > 8$		$3x < -8$
		$x < -\frac{8}{3}$

The above solution can be expressed as follows.

Set Notation: $\{\, x \mid x > 8 \; or \; x < -\frac{8}{3} \,\}$ **.**

Interval Notation: $(-\infty, -\frac{8}{3}\,) \cup (\,8, \infty)$ **.**

Graphically on a Number Line: