4.4 Exponential Growth and Decay

We complete this course with one of the most versatile concepts in pre-calculus: exponential growth and decay. The usefulness of this concept is limitless. No other function outlined in this text compares to the usefulness of exponential growth and decay, since this concept is used to model events that occur in the natural world. Our environment is constantly changing. Calculus is used to describe the rate at which things change, and a great deal more. In this section we will not explicitly use calculus, but we will examine in detail one of it's many important consequences.

Exponential growth and decay ocurrs when something changes in proportion to its current value. Some bacteria for example will triple in population every two minutes if left to grow uninhibited. Antibiotics are used to fight and eventually kill some bacteria. But without antibiotics, many people would die from bacterial infections. We are seeing this now in hospitals all over the world. People check into a hospital with a non-lethal injury and end up dying from a simple staph infection. This is because some bacteria has evolved and learned to adapt and overcome the effects of antibiotics. The point here is that when things are left to grow or decay uninhibited in proportion to their current population, exponential growth and decay occurs, and it is happening a lot more than you might think. From the decay of Carbon-14 in dying organic matter to the decay of byproducts from nuclear fission, exponential growth and decay is an integrable part of our ever-changing world.

Although this simple equation can be written in many different ways when discussing different phenomena, we will stick to just the one form for the sake of simplicity.

$y(t) = y_o e^{k \cdot t}$

At first glance, as with most everything in mathematics, this equation may seem intimidating. But as we will soon see, it is in fact a very straightforward and simple concept to master. First, it is important to know exactly what each variable means. Once we understand this, we can set up a three-part algorithm for solving any exponential growth and decay problem. Not only is this one of the most useful equations in mathematics, with a little practice and the correct guidance, it may be one of the easiest to understand.

The independent variable t in this equation represents the appropriate unit of time.

When the phenomenon begins, we let $t = 0$ and then $y(0) = y_o e^{k \cdot 0}$. Since $e^0 = 1$ we have that $y(0) = y_o$.

This term y_o is called "y-naught" which literally means "y of nothing." This value y_o is our initial amount. It is the quantity that is originally given. That is, the amount that you start with in the beginning when $t = 0$.

The term k is the growth constant and is different for each problem. It is also what we need to find first when formulating an equation for a specific phenomenon. It is important to note that when $k > 0$ we have growth, and when $k < 0$ we have decay.

Nearly any question regarding an exponential growth and decay problem can be solved by understanding three basic sub-problems. The mechanics of these problems never change. Let us start with some basic problems and from there we can formulate a universal algorithm.

Example 1: Recall the scenario stated at the beginning of this section. Suppose that a given bacteria population triples every two minutes. Before we can even discuss this phenomenon, we need to create a formula specific to the situation. Remember that the first step in doing this is solving for the growth constant $k > 0$ (the constant is greater than zero because we have growth). For the sake of argument, let us suppose we start with 10 mg of bacteria. Later we will see that this has no real bearing on the first step.

First we take what we know.

Call this Part A.

$y(2) = 10e^{k \cdot 2}$ **and** $10e^{k \cdot 2} = 30$

$e^{k \cdot 2} = 3$

$LN(e^{k \cdot 2}) = LN(3)$

$k \cdot 2 = LN(3)$

$$k = \frac{LN(3)}{2}$$

$y(t) = 10 \cdot e^{\frac{LN(3)}{2} \cdot t}$**mg**

Finding the equation that explains this phenomenon at any $t > 0$ must always be the first step, unless of course the equation is already given.

If we were now asked "How many bacteria will there be in 4 hours?" we would simply convert to minutes (the original unit of time in which the problem was given) and plug that number in for t.

Here 4 hours is $4 \cdot 60 = 240$ minutes.

Call this Part B.

So we have that $y(240) = 10 \cdot e^{\frac{LN(3)}{2} \cdot 240}$mg, which can be simplified as follows.

$y(240) = 10 \cdot (e^{LN(3)})^{\frac{240}{2}} = 10 \cdot 3^{120}$

$= 17,970,102,999,144,312,104,131,798,295,096,050,397,314,756,275,378,511,064,010$ **mg**

Finally, we can ask *how long* it takes to achieve a certain amount.

Call this Part C.

How long will it take to achieve $100,000$ mg?

Here we simply set the original equation equal to the desired amount and solve for t.

Just as in Part A, the algorithm for solving this problem will always be the same.

Let $10 \cdot e^{\frac{LN(3)}{2} \cdot t} = 100,000$

$e^{\frac{LN(3)}{2} \cdot t} = 10,000$

$LN(e^{\frac{LN(3)}{2} \cdot t}) = LN(10,000)$

$\frac{LN(3)}{2} \cdot t = LN(10,000)$

$t = \frac{2}{LN(3)} \cdot LN(10,000)$ **minutes, or** 16.767 **minutes.**

In this section not only will we see the wide range of problems we can solve by exponential growth and decay, but how every problem, no matter how unrelated, is solved with the same method and tempo. It is basically the idea of this entire text summed up into one mathematical concept.

The previous example was one of exponential growth, that is where $k > 0$. Here let us explore exponential decay, where $k < 0$. Notice the similarities between this example and the previous one. The only real difference are the values of each constant. Focus on the method and the similarities herein.

Example 2: Polonium-210 is a highly radioactive isotope with a half-life of 138 days. Note that 210 is the name of the isotope and has no mathematical bearing on the problem. Moreover, exponential decay is always measured as a half-life. In this case, after 138 days only half of the amount of the original substance will remain.

Part A. Suppose we begin with a sample of 40 grams of Polonium-210. Find an equation for the decay of Polonium-210 after t days.

We begin with the information we are given just as we did in the last example. The first step in Part A is to solve for k.

$$40 \cdot e^{k \cdot 138} = 20$$

$$e^{k \cdot 138} = \frac{1}{2}$$

$$LN(e^{k \cdot 138}) = LN(\frac{1}{2})$$

$$k \cdot 138 = LN(\frac{1}{2})$$

$$k = \frac{LN(\frac{1}{2})}{138}$$

It is important to notice that since this is exponential decay, we must have that $k < 0$. If we recall the second of the Three Laws of Logarithms $LN(\frac{1}{2}) = LN(1) - LN(2) = -LN(2)$, as $LN(1) = 0$.

Also note that k being the growth/decay rate, it is always $\frac{LN(a)}{b}$ where a is the amount of change that occurs over the period of time b. Ideas like this are good to notice so that we know when we are on the right track. They help us maintain our confidence when working through the method. They are not meant as a means to skipping steps. When we skip steps, we are no longer practicing mathematics.

The final equation for the decay of 40 grams of Polonium-210 is $y(t) = 40 \cdot e^{\cdot \frac{-LN(2)}{138} \cdot t}$ grams.

Part B. How much of the 40 grams Polonium-210 will remain after one year?

Since one year is 365 days, we have that $y(365) = 40 \cdot e^{\cdot \frac{-LN(2)}{138} \cdot 365}$ grams or 3.695 grams.

Part C. How long will it take for 40 grams of Polonium-210 to be reduced to 1 mg?

Since 1 mg = 0.001 grams, we have $40 \cdot e^{\frac{-LN(2)}{138} \cdot t} = 0.001$.

Now solve for t.

$e^{\frac{-LN(2)}{138} \cdot t} = \frac{0.001}{40}$

$LN(e^{\frac{-LN(2)}{138} \cdot t}) = LN(\frac{0.001}{40})$

$\frac{-LN(2)}{138} \cdot t = LN(\frac{0.001}{40})$

$t = \frac{138}{-LN(2)} \cdot LN(\frac{0.001}{40})$ **days or** 2109.70 **days.**

As mentioned numerous times throughout this text, it is necessary to write and re-write each of the examples several times. In doing so, we find that the steps are identical. We will find a rhythm, and our performance will become second nature. Remember that achieving a conditional response (or a second-nature response) is the desired effect in practicing mathematics.

Interesting Fact: Polonium-210 was discovered by the chemist and physicist Marie Sklodowska Curie in 1898. As a French immigrant, Curie named her discovery after her native born country of Poland. Marie Curie was the first woman to win the Nobel Prize and first person to win it twice, and in two different fields: chemistry and physics, in 1911 and 1903, respectively. She died in 1934 from aplastic anemia due to exposure to radioactive radium which she carried in her lab coat on several occasions.

Example 3: Population growth of any organism may be modeled using exponential growth. In the natural world there are nearly always limiting factors that make total uninhibited growth impossible. For example, coyotes eat rabbits and keep their population in check, as rabbits are well known for breeding out of control. The same is true with humans. Before 1900, human population never exceeded one billion people. After 1900, human population exploded and started experiencing (limited) uninhibited growth. There are many factors that made this possible. Antibiotics were first developed. In particular, bacteria-fighting molds like penicillin. Vaccines were being developed one after the other circa 1900. And of course, the potential energy and endless applications of fossil fuels were first realized. At the end of 2016, human population was just shy of 7.5 billion. This shows that humans began experiencing uninhibited growth around 1900. Let us analyze this growth and see what we can learn by applying the same methods as we did in the previous two problems.

Part A. Find an equation that models human population growth beginning in 1900. Try to follow the methods used in the previous two examples. First we use what we are given and solve for k. Notice that we are starting at 1900, so we let $t = 0$ at 1900. For our units, we will choose "billion people," so our initial value $y_o = 1$ *billion people.*

Now let $t = 116$ for the year 2016. Since 2016-1900=116.

$e^{k\cdot 116} = 7.5$

$LN(e^{k\cdot 116}) = LN(7.5)$

$k\cdot 116 = LN(7.5)$

$k = \frac{LN(7.5)}{116}$

$\therefore y(t) = e^{\frac{LN(7.5)}{116}\cdot t}$ **billion people.**

Part B. Given this trend in world human population growth, use the equation in Part A to estimate human world population in 20 years.

In 20 years we will have that $t = 136$.

The rest is simple. Just plug in this value for t.

$y(136) = e^{\frac{LN(7.5)}{116} \cdot 136} = 10.61533966$ **billion people.**

For curiosity's sake, let us estimate what the population will be in 2100.

$y(200) = e^{\frac{LN(7.5)}{116} \cdot 200} = 32.26460229$ **billion people.**

By the end of the century, if we continue this growth rate, there will be about 4.3 times as many people on earth than there are today. The question is, can the earth sustain this many people? Many scientist believe that where food is concerned, the earth can only sustain 10 billion people. As seen in Part B, this may occur within the next 20 years.

Part C. When will the human population in 2016 double?

Here we set our equation from Part A equal to 15 and solve for t.

$e^{\frac{LN(7.5)}{116} \cdot t} = 15$

$LN(e^{\frac{LN(7.5)}{116} \cdot t}) = LN(15)$

$\frac{LN(7.5)}{116} \cdot t = LN(15)$

$t = LN(15) \cdot \frac{116}{LN(7.5)} =$**155.9 or 156, which will be in 2056.**

Example 4: Now we return to exponential decay. All organic material contains Carbon-14, which has a half-life of 5730 years. Radio carbon dating actually uses the fact that when organic material dies, it begins to release Carbon-14. Using exponential decay, we can find out the age of something that died long ago. Suppose we find a fossilized tree branch that contains 10% of the level of Carbon-14 that it had when it died. Can we find out how old the tree branch is?

Here we are not going to bother going thorough parts A, B, and C. However, we are going to use the knowledge we have gained through practicing these steps to find the age of the branch.

We of course need to start with an equation, which is usually what we refer to as Part A.

$y_o \cdot e^{k \cdot 5730} = 0.5 \cdot y_o$

$e^{k \cdot 5730} = 0.5$

$LN(e^{k \cdot 5730}) = LN(0.5)$

$k \cdot 5730 = LN(0.5)$

$k = \frac{LN(0.5)}{5730}$

Here $y(t) = y_o \cdot e^{\frac{LN(0.5)}{5730} \cdot t}$ **Carbon-14 remains after** t **years.**

That was essentially Part A. When discussing these phenomenon you always need to create an equation that accurately demonstrates exponential growth or decay.

Now we are asking how much Carbon-14 we will have after a certain time, so we do not need to discuss Part B.

However, the question is "when." When will the Carbon-14 be reduced to 10% of the original amount? Now invoke the method applied in Part C, and we are done.

$y_o \cdot e^{\frac{LN(0.5)}{5730} \cdot t} = 0.1 \cdot y_o$

$e^{\frac{LN(0.5)}{5730} \cdot t} = 0.1$

$LN(e^{\frac{LN(0.5)}{5730} \cdot t}) = LN(0.1)$

$\frac{LN(0.5)}{5730} \cdot t = LN(0.1)$

$t = \frac{5730}{LN(0.5)} \cdot LN(0.1)$ **years old.**

Or $t = 19034.64798$**, rounded to** $t = 19,035$ **years old.**

Now that we have discussed some of the more common uses for exponential growth and decay, we shall look at just how versatile this equation can be.

Example 5: In the United States, it is illegal to operate a motor vehicle with a blood alcohol content (BAC) of 0.08% or higher, although a driver may become impaired and risk crashing at a BAC as low as 0.02%. In fact, in Sweden it is illegal to drive with a BAC of 0.02% or higher, and the punishment calls for up to six months in prison. Alcohol is excreted by the body in a variety of ways over time and can be modeled using exponential decay models. As well, the risk of crashing increases exponentially as BAC increases. So both phenomena can be interpreted using the equation $y(t) = y_o e^{k \cdot t}$**.**

According to the DUI Justice Link, a person between the ages of 16 and 20 has a 4.09 times greater risk of crashing with a BAC of 0.05% and is 9.51 times more likely to crash with a BAC of 0.08%. This clearly shows that the risk of crashing increases exponentially with an increased BAC.

Part A. Given the information here, let us write a clear formula for the risk of crashing an automobile, for the age group of 16-20, given any particular BAC. Remember to approach this just as we approached the other problems. Not much will change. Instead of an independent variable of time, we will now use BAC represented by x**. Now we give it a try using the information we are provided.**

$y_o \cdot e^{k \cdot (0.05)} = 4.09 \cdot y_o$

Here the value of y_0 **is not really a factor, since a sober driver with a BAC of 0.00% is 1 times as likely to crash than another sober driver. We write it down initially out of habit, which is good, but after we start running the calculations and give it some thought, it is clear that** $y_o = 1$**.**

$e^{k \cdot (0.05)} = 4.09$

$LN(e^{k \cdot (0.05)}) = LN(4.09)$

$k \cdot (0.05) = LN(4.09)$

$k = \frac{LN(4.09)}{0.05}$

$R(x) = e^{\frac{LN(4.09)}{0.05} \cdot x}$, **which we can simplify to** $R(x) = e^{28.17 \cdot x}$, **where** $R(x)$ **is risk and** x **is BAC.**

Part B. Use the equation found in Part A to verify the risk assessment for a BAC of 0.08% being 9.51 according to the DUI Justice Link.

$R(0.08) = e^{28.17 \cdot (0.08)}$ **and so** $R(0.08) = 9.52$

This shows that there is an exponential correlation and our equation is within acceptable parameters.

Part C. What would the driver's BAC have to be for the risk of crashing to be 100 **times more likely to than a sober driver. It is worthwhile to take stock of this for a moment. When a person is 100 times more likely to crash, they will most certainly crash. Think about it for a moment and compare it to something you can relate to. If you were 100 times more likely to die or possibly kill someone, would you take the risk? If you were 100 times more likely to win a lottery, would you make that bet? There are many ways to think about this and many analogies we could make. But it all comes down to the fact that a very bad outcome is 100 times more likely than usual, and those are extremely poor odds.**

$e^{28.17 \cdot x} = 100$

$LN(e^{28.17 \cdot x}) = LN(100)$

$28.17 \cdot x = LN(100)$

$x = \frac{LN(100)}{28.17}$ **or approximately** $\%0.16$ **, which is only twice the legal limit.**

In 2014, approximately one person died every 53 minutes in the United States as a result of alcohol-impaired driving.[1]

[1] http://duijusticelink.aaa.com/facts/