4.3 Solving Exponential and Logarithmic Equations

It often seems, when doing mathematics, that there is an infinite number of ways by which to solve a single set of problems. This may become overwhelmingly daunting. With practice and the correct direction, mathematics may become quite simple. The truth is that every one at some point struggles with mathematics in their own unique way. Some learn the lesson set forth in this manual early, others never find the way. Effective practice is done by achieving object recognition and conditional response. Categorize the type of problem through object recognition and then be confident in how to deal with it through a trained conditional response. This is the basic idea that has been repeated throughout this text. It is a highly effective technique if implemented correctly. However, it is a necessary condition of this technique that the student practice with persistence and not choose to give up.

In the previous section we found that the following is true when the bases are the same.

$5^{x+2} = 5^{7x} \Longrightarrow log_5(5^{x+2}) = log_5(5^{7x}) \Longrightarrow x + 2 = 7x \Longrightarrow 2 = 6x \Longrightarrow \frac{1}{3} = x$

Now how do we solve an exponential equation with two different bases? To answer this we use the properties of the Natural Logarithm and the Three Laws of Logs.

First we need to categorize the problem in order to find the most suitable algorithm for addressing the issue. Remember that mathematics is just a game with a very short list of rules.

Example 1. Solve the following for x**.**

$6 = 3 \cdot e^{-x}$

The first step in solving an equation with an exponential function is to have a single base on one or either side. In this case the task is simple. We just move the 3 **from the right side of the equation to the left. Remember that there is always more than one way to solve a problem. However, it is best to stick to the same method every time when that method is effective. Repetition is the key.**

Now $2 = e^{-x}$**.**

Remember that bringing the exponent down is not difficult. Simply apply Law Three of the Three Laws of Logs. Of course the most versatile log to use is the natural logarithm, which is more than appropriate here. The natural logarithm is $log_e(x) = LN(x)$ and the inverse function of e^x.

$LN(2) = LN(e^{-x)}$

$LN(2) = -x$

$x = -LN(2)$

For the next example we use the same method, but we have designed it to appear more intimidating. If we have trouble understanding the movements, we simply repeat them as many times as it takes to understand the mechanics.

Example 2: Solve for x.

Note that both sides are exponents, and we will need to take logs of either side.

$7^{x+1} = 3^{2x-5}$

$LN(7^{x+1}) = LN(3^{2x-5})$

$(x+1)LN(7) = (2x-5)LN(3)$

$xLN(7) + LN(7) = 2xLN(3) - 5LN(3)$

$xLN(7) - 2xLN(3) = -5LN(3) - LN(7)$

$x(LN(7) - 2LN(3)) = -5LN(3) - LN(7)$

$$x = \frac{-5LN(3) - LN(7)}{LN(7) - 2LN(3)}$$

We could do several examples like this, but just as with many other problems there is one simple algorithm that will yield the desired result. One must concentrate on recognizing the problem and practicing the preferred method for solving that problem. No matter what the bases, as long as we have a function of x in the exponent, the above method will work. Focus on the steps just as we would any other methodical or repetitive task, like baking a cake or riding a bicycle.

The final result may look strange and it is tempting to simplify. However, the student should not attempt this. If a decimal number solution is required, then use a calculator. Most simple scientific calculators can evaluate $LN(x)$ and $log_{10}(x)$, which is why we often use $LN(x)$ or $log_{10}(x)$ when attempting to solve exponential equations with different bases.

This leads us to the change of base formula. When we have a logarithm with an undesirable base, we can always change the base as follows.

$log_a(x) = \dfrac{log_b(x)}{log_b(a)}$ **where $a \neq 1$,$b \neq 1$ and x is a positive real number.**

Now we will toggle back and forth between logarithmic and exponential equations.

Notice that there are more logarithm examples than there are exponential examples in this section. This is because we are more use to manipulating exponents than logs.

When solving logarithmic equations, it is ideal to have one log on one side and a number on the other or have both the left and right sides contained within a single log on each side. Let us start with an example of one log on one side of the equation and a single constant on the other. Since these problems are so unfamiliar to most, try to notice the steps taken and not just the beginning and end results. Remember that the point of a sentence is not in the period at the end, but in the content conveyed within the structure of the sentence.

Example 3: Solve for x.

Note that this exercise involves logs, so we need to take exponents.

$log_2(2x - 1) = 3$

$2^{log_2(2x-1)} = 2^3$

$2x - 1 = 8$

$2x = 9$

$x = \frac{9}{2}$

Notice that this next problem looks much like the last one. It is more beneficial to look for the similarities and not the differences. If we concentrate on the similarities, the differences will become known. The procedure for solving both problems is the same. This is the key to understanding how to solve each and every problem. There are not many different ways to do this, which is a main idea of the course.

Example 4: Solve for x.

$log_3(4x - 7) = 2$

$3^{log_3(4x-7)} = 3^2$

$4x - 7 = 9$

$4x = 9 + 7$

$4x = 16$

$x = 4$

Example 5: Solve for x.

$log_5(2x + 3) = log_5(7)$

This is another ideal situation because we have one log on one side and one log on the other. Both are also the same base, which is helpful.

Since the base of the logarithm is 5, we put both sides in the power of the base 5 exponent.

$5^{log_5(2x+3)} = 5^{log_5(7)}$

The exponent-base 5 and the logarithm-base 5 are inverse functions, so they unravel each other and we have the following.

$2x + 3 = 7$

$2x = 4$

$x = 2$

When we finally solve for x given a logarithmic function, we must remember that the logarithm has a restricted domain and that we can not put zero or negative numbers inside a logarithmic function. We return to the original problem, plug in our newfound value for x, and check that it does not create zero or a negative input for the logarithm.

Notice that the above two examples have only one solution and do not utilize any of the Three Laws of Logs. This next exercise will take it a step further, with two possible solutions.

Keep in mind that it does not matter if the logarithms are added or subtracted from either side, as that can be easily remedied with the Three Laws. Yet we still desire a single log on either side, or one log on one side and a constant on the other.

Example 6: Solve for x.

$log_2(x + 7) + log_2(x + 8) = 1$

By applying Law 1 we have,

$log_2((x + 7)(x + 8)) = 1$

$log_2(x^2 + 15x + 56) = 1$

Raising both logs to the power of 2 yields.

$2^{log_2(x^2+15x+56)} = 2^1$

$x^2 + 15x + 56 = 2$

$x^2 + 15x + 54 = 0$

$(x+6)(x+9)=0$

So $x=-6$ and $x=-9$.

Now recall that the logarithm can only take non-zero positive numbers.

We return to the original equation and let $x=-6$,

then $log_2(-6+7)+log_2(-6+8)=1$

$log_2(1)+log_2(2)=1$

$0+1=1$ **which is true.**

The solution $x=-6$ checks out.

However, when we let $x=-9$

$log_2(-9+7)+log_2(-9+8)=1$

$log_2(-2)+log_2(-1)=1$

The input values for both logarithms are not allowed. They are not in the domain of the logarithmic function. They do not exist.

There is only one valid solution.

WARNING! When solving logarithms, we must return to the original equation because the logarithm has a restricted domain and can take only non-zero positive values.

Recall that the domain of a function is the range of the inverse function.

Because $D_f=\{x \mid x>0\}$, when $f(x)=log_2(x)$, then $R_{f^{-1}}=\{y \mid y>0\}$, when $f^{-1}(x)=2^x$.

When the domain is not restricted, we do not need to check the result and validate the solution. However, there is a special situation when we need to validate the solution with the range. This occurs we when use a substitution method.

Example 7: Solve for x.

$9^x + 3^x - 2 = 0$

Now the Three Laws of Logs do not seem to apply here, because none of the laws involve a sum or difference inside the logarithm.

Look closer. This is an idea outlined throughout this text.

What if we write it a bit differently?

$3^{2x} + 3^x - 2 = 0$ **or** $(3^x)^2 + (3^x) - 2 = 0$

Replace the exponential term with a "*dummy*" variable. That is to say, use substitution.

Let $3^x = t$ **so now the above equation can be expressed as a quadratic.**

$t^2 + t - 2 = 0$

$(t + 2)(t - 1) = 0$

$t = -2,\ 1$

But $3^x = t$

Now we check.

If $t = 1$**, then** $3^x = 1$ **and therefore** $x = 0$**.**

If $t = -2$**, then** $3^x = -2$**.**

But negative numbers are not in the range of exponential functions, as they are NOT in the domain of logarithmic function. The second scenario with $t = -2$ **can not yield a solution.**