4.1 The Exponential Function

Definition 1: An *Exponentiation Function* is a function having independent variables expressed in the exponent. The general form of the exponential function may be written as $f(x) = a^x$, where a is a constant known as the "*base*" of the function and x is the "*exponent.*"

Euler's Number, named after the Swiss mathematician Leonard Euler, is of particular importance when discussing the exponential function. A discussion of the significance of Euler's number is not necessary at this point, but it is of great importance in the subject of exponential growth and decay. Euler's Number is known as a transcendental number (e.g., π) and is defined as follows:

$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots \approx 2.71828...$ **and** $e^x = \left(1 + \frac{x}{n}\right)^n$ **if** n **is taken to** ∞**.**

Exponential functions can be tricky at first. This is only because they are unfamiliar to the pre-calculus student. Their mechanics seem precarious and they often invoke a xenophobic reaction. This may be intimidating, like trying a new sport or picking up a musical instrument for the first time. The good news is that mathematics requires no unique physical abilities. It simply requires the memorization of a limited number of rules and hard work. When it comes to exponential functions, there are only five laws that govern their behavior. Memorize these until you can visualize them when you close your eyes and know that one side of the equation is the exact same as the opposite side. This is more crucial than it sounds.

Memorize the following laws by rewriting them perfectly. Do not change the order or the variables in their present form. For each law we assume that $a,\ b > 0$. When writing and re-writing each law you must focus and repeat what each equation is expressing in mathematical terms, like a mantra. Even tempo is important. This practice will reinforce the conditional response, which is the desired effect.

THE FIVE LAWS OF EXPONENTS

$$a^x \cdot a^y = a^{x+y}$$

$$(a^x)^y = a^{x \cdot y}$$

$$\sqrt[y]{a^x} = a^{\frac{x}{y}}$$

$$a^{-x} = \frac{1}{a^x} \text{ or } \frac{1}{b^{-x}} = b^x$$

$$a^x \cdot b^x = (a \cdot b)^x$$

Each equation should be written at least five times (more if necessary). Focus on the fact that each side of the equation has the same meaning. As you copy each, repeat helpful statements about each that will aid you in understanding their basic meaning.

For example, when writing the first law repeat aloud to yourself "multiply bases - add exponents." Remember that no matter how silly it sounds, this is a highly effective technique. This is even more effective when repeated in groups. Repeat each mantra with the corresponding law.

1. "Multiply bases - add exponents."

2. "The exponent of an exponent - multiply exponents."

3. "The root is the rational exponent."

4. "The negative exponent means flip the base of the exponent."

5. "Bases with the same exponent may be multiplied and collected into a single exponent."

Basic Properties of the Exponential Function

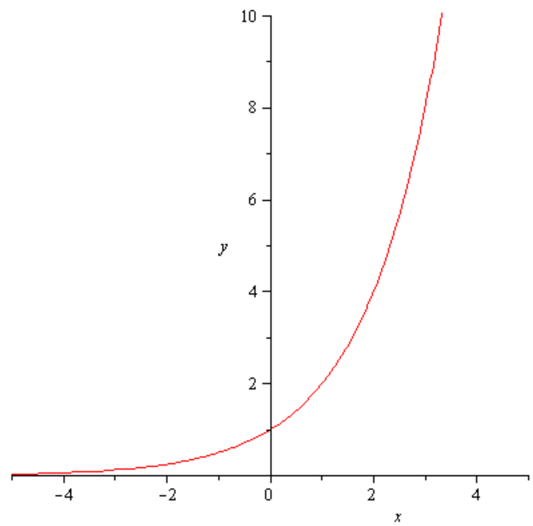
Recall that one of the most fundamental properties of a function is the domain: the set of all possible inputs. Since $f(x) = a^x$ is not among our list of three cautionary functions, the domain is the set of all real numbers. Written in set notation as $D_f = \{x \| x \in \mathbb{R}\}$. The exponential function is one-to-one and has an inverse. This is the fogarithmic function and it is written as $f^{-1}(x) = \log_a(x)$. Read as "*log-base-a of x*." The domain of the logarithmic function is the range (the set of all possible outputs) of the exponential function. The logarithmic function has a restricted domain and can only take positive numbers. Thus the domain of the logarithmic function is written in set notation as $D_{f^{-1}} = \{x \| x > 0\}$. It is important that we remind ourselves that inverse function 0 simply switches inputs and outputs therefore, the range of one is the domain of the other, and visa versa.

The concept of logarithmic functions is simple. Remember that the inverse function undoes the work of the original function. There are basically three tiers of arithmetic operations: addition-subtraction, multiplication-division, exponents-logarithms. Each are inverse operations of the other. Do not overthink the logarithmic function. When you think of the logarithm, think of it as the function that unravels the x from the exponential function. It is much like addition undoing subtraction, or multiplication undoing division.

$f(x)$	$f^{-1}(x)$	$f(f^{-1}(x)) = x$	$f^{-1}(f(x)) = x$
$x + 3$	$x - 3$	$(x - 3) + 3 = x$	$(x + 3) - 3 = x$
$5.x$	$\frac{1}{5}.x$	$5.(\frac{1}{5}.x) = x$	$\frac{1}{5}.(5.x) = x$
7^x	$log_7(x)$	$7^{log_7(x)} = x$	$log_7(7^x) = x$

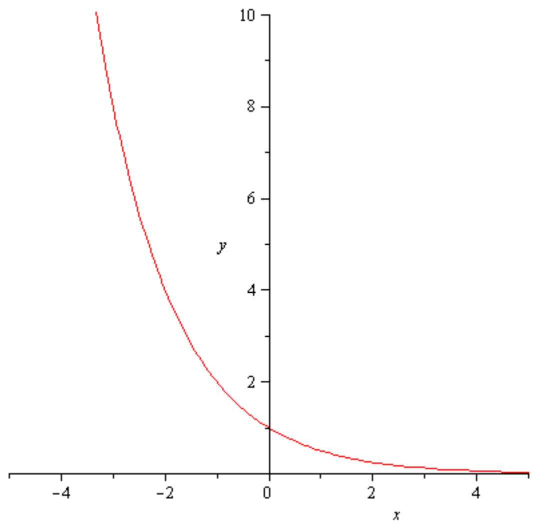
Example 1: Graph the function $f(x) = 2^x$.

It is important to note that the function passes through three significant points $\left(-1,\ \frac{1}{2}\right), (0,\ 1), (1,\ 2)$:



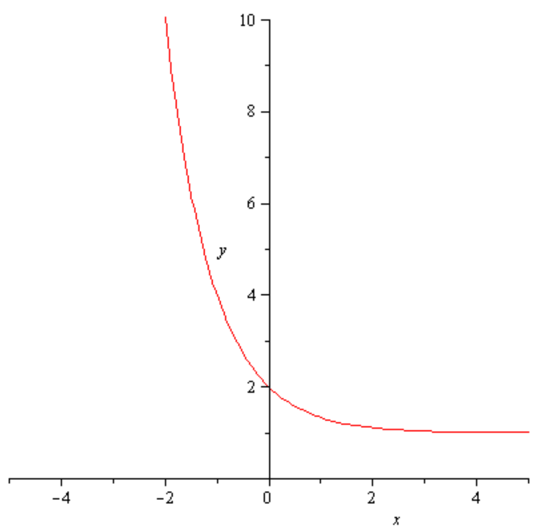
Example 2: Graph the function $f(x) = 2^{-x}$.

It is important to note that the function passes through three significant points $(-1,\ 2), (0,\ 1), (1,\ \frac{1}{2})$ and that $f(x) = (\frac{1}{2})^x$as $2^{-x} = (\frac{1}{2})^x$.



Example 3: Graph the function $f(x) = 3^{-x} + 1$.

It is important to notice that the function passes through three significant points $(-1,\ 4), (0,\ 2), (1,\ \frac{4}{3})$.



Example 4: Solve the following equation for x. $27^x = \frac{1}{81}^{-x+1}$

Later we will show that if $a^x = a^y$ then $x = y$. For now we will assume that it is a true statement, and all we need to do is express each side of the equation as a power of the same base. Notice that both 27 and 81 can be expressed as a power of 3. Now just apply the Five Laws of Exponents as follows.

$(3^3)^x = (\frac{1}{3^4})^{-x+1}$

$3^{3x} = (3^{-4})^{-x+1}$

$3^{3x} = 3^{4x-4}$

$3x = 4x - 4$

$-x = -4$

$x = 4$

Example 5: Solve the following equation for x.

$9^x - 2 \cdot 3^x = 3$

$9^x - 2 \cdot 3^x - 3 = 0$

$t = 3^x$

$3^{2x} - 2 \cdot 3^x - 3 = 0$

$t^2 - 2 \cdot t - 3 = 0$

$(t-3) \cdot$ **(t+1)=0**

$t = 3, -1$

$3^x = 3$ **so** $x = 1$

$3^x = -1$ **which has no solution.**

Therefore $x = 1$ is the only solution.

Example 6: A classic exponential growth problem.

Suppose you are walking along the Pacific Crest Trail and you encounter a genie. The genie offers you a choice. You may either have \$1,000,000 cash or you may have a magic penny. Every day each existing penny will duplicate and yield two total magic pennies from the one. This will continue for 40 days. What should you choose? The cash in hand, or the magic penny?

Answer: Choose the magic penny. Here is why.

On the day the genie gives you the penny, call it "day zero," $t = 0$ **and you have just one penny. On day one, when** $t = 1$**, you have two pennies. This continues and will accumulate as follows:** $1, 2, 4, 8, 16, 32...$ **and so on. On day** 40 **you will have** $2^{40} = 1099511627776$ **pennies, which is \$10,995,116,277.76.**

This shows just how fast the exponential function may grow.