3.4 Polynomial and Rational Inequalities

Recall that inequalities are no different than equalities, except when we multiply or divide by a negative number we must flip the inequality. We will not see much of that in this section. However, we will be solving polynomials and rational functions and as usual we will want to get every term on one side on the inequality and zero of the other. It is tempting to cancel terms, but remember that when you cancel terms that include a variable you are essentially dividing by an infinite number of unknowns that will most likely include zero.

Since we have covered most of the tools that we need to use in this section in the previous sections of Chapter 3, it is best if we get started right away with some examples.

Example 1: Solve $x < x^3$ **and graph the solution set. Be sure to write your solution in interval notation.**

First we move all terms to one side so that 0 **is the only term on one side.**

$0 < x^3 - x$

Remember what is being asked here.

Where is the cubic polynomial $x^3 - x$ **strictly above the** ***x-axis*****?**

Where is it always positive and not zero? Now continue by factoring.

$0 < x(x^2 - 1)$

$0 < x(x + 1)(x - 1)$

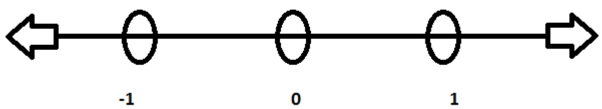
A polynomial changes from positive to negative is at zero. We set each of our terms to zero and solve.

$x = 0$

$x + 1 = 0 \Rightarrow x = -1$

$x - 1 = 0 \Rightarrow x = 1$

This gives us three roots, which means that there are three places where the polynomial can change from positive to negative or negative to positive.



Now give this inequality a name. Say $f(x) = x^3 - x = x(x+1)(x-1)$, and find where $f(x) > 0$.

That is, where is it positive (above the ***x-axis***), but not equal to zero (not touching the ***x-axis***.) The easiest way to do this, and the same way we will want to do it in calculus, is to do a number-line analysis, as shown above.

The points at $x = -1, 0, 1$ are not filled in because they are not included in the solution set.

Now everything on the left of -1 will be positive or negative when plugged into $f(x)$.

This will also be the same for all numbers between -1 and 0, 0 and 1, and everything to the right of 1.

This is because $f(x)$ can only change signs at those points.

In other words, $f(-1) = f(0) = f(1) = 0$, and can only change from positive to negative or negative to positive at these points.

Now check points in between -1, 0, and 1, in $f(x)$ to see if they are positive or negative. If they are positive, then these are the intervals we want.

Choose some random point on the left of -1.

How about -10? Then $f(-10) = (-10)^3 - (-10) = -1000 + 10$. This number is clearly negative.

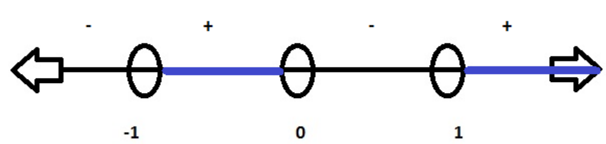
Now we check a number between $x = -1$ and $x = 0$. How about$x = -\frac{1}{2}$?

Then $f(-\frac{1}{2}) = (-\frac{1}{2})^3 - (-\frac{1}{2}) = -\frac{1}{8} + \frac{1}{2}$. This number is clearly positive.

We continue, checking $f(\frac{1}{2}) = (\frac{1}{2})^3 - \frac{1}{2}$, which is negative.

Finally, check to the right of 1 with $f(10) = (10)^3 - 10 = 1000 - 10 > 0$, again positive.

Hence our solution for $f(x) > 0$ is on the intervals $(1, 0)$ and $(1, \infty)$. The graph of our solution set is shown in blue below.



There are several variations of this problem.

What if $f(x) \geqslant 0$? Then the solution set would include the points where $f(x)$ touches the graph and the solution set would be $[-1, 0]$ and $[1, \infty)$.

The circles on the graph at -1 and 1 would also be filled in. Notice that ∞ cannot be enclosed with a bracket or included as a point in the set. This is why we use parentheses.

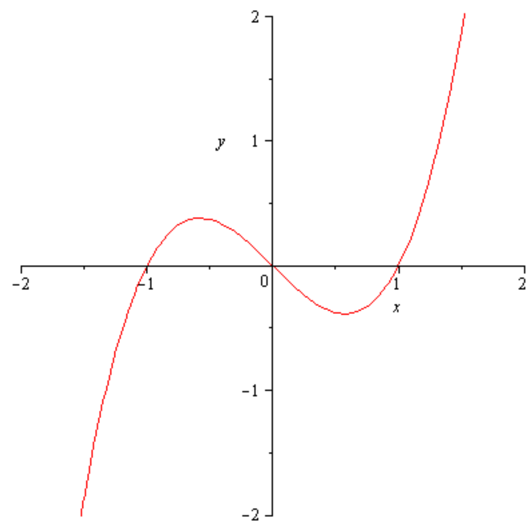
If $f(x) < 0$ were the problem we were trying to solve, it would be just the opposite of the first problem.

The solution set would be $(-\infty, -1)$ and $(0, 1)$ and all of the circles would be open. Can you find the graph?

Now if you wanted to solve $f(x) \leq 0$, simply include the end points from the last solution.

Then the solution set would be $(-\infty, -1]$ and $[0, 1]$. The circles at -1, 0, and 1 would be filled in on the graph.

To gain better insight, look at the graph of $f(x) = x^3 - x$.



Example 2: Solve $(x+3)(x-1)^2 \geq 0$ and graph the solution set. Be sure to write your solution in interval notation.

First call $f(x) = (x+3)(x-1)^2$.

Note that $x = -3, 1$ are the points where $f(x) = 0$.

These are exactly the points where the function $f(x)$ changes from positive to negative.

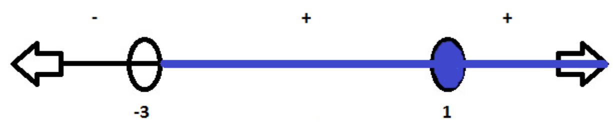
Also notice that the multiplicity of -3 is one, which is odd, and 1 is two, which is even.

Next we do a number-line analysis and check the intervals before and after -3, and before and after 1.

$f(-4) = (-1)(-5)^2 = -25 < 0$

$f(0) = (3)(-1)^2 = 3 > 0$

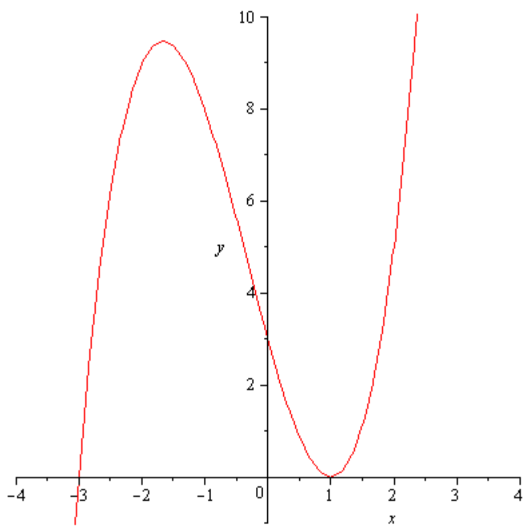
$f(2) = (5)(1)^2 = 5 > 0$



The solution set includes 1**, so in interval notation we have** $(-3, \infty)$.

Now try to find the solution sets for when $f(x) > 0$, $f(x) < 0$, **and** $f(x) \leq 0$.

The graph of $f(x) = (x+3)(x-1)^2$ **is shown below.**



Here we change gears and take a look at rational inequalities. The one thing to remember with rational inequalities is that when the bottom is equal to zero, the rational function does not exist. Therefore, the points not included in the domain can not be included in the solution set because they do not exist.

Example 3: Solve $\dfrac{5x-3}{x+1} \geq 4$ **and graph the solution set.**

Be sure to write your solution in interval notation.

First we need to get zero on one side.

$\frac{5x-3}{x+1} - 4 \geqslant 0$

Just like any other fraction, to add the two together we need a common denominator.

$\frac{5x-3}{x+1} - \frac{4(x+1)}{x+1} \geqslant 0$

$\frac{5x-3-4(x+1)}{x+1} \geqslant 0$

$\frac{5x-3-4x-4}{x+1} \geqslant 0$

$\frac{x-7}{x+1} \geqslant 0$

As before, give it a name. Let $f(x) = \frac{x-7}{x+1}$**.**

Now remember that $x \neq -1$**, and it can never be included in the solution set.**

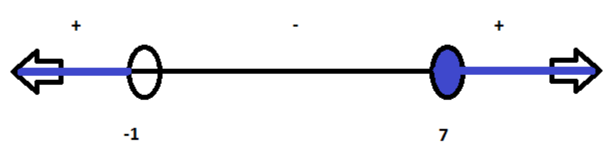
Nevertheless, when we apply the number line analysis we check the intervals around -1 **and** 7**.**

$f(-2) = 9 > 0$

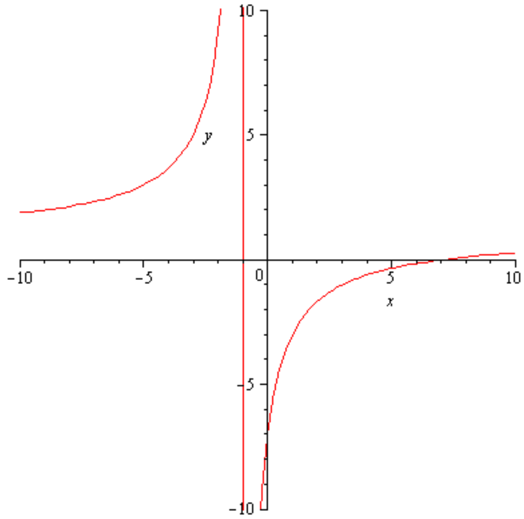
$f(0) = -7 < 0$

$f(8) = \frac{1}{9} > 0$

The solution set is $(-\infty, -1) \cup [7, \infty)$**.**



To gain better insight, look at the graph of $f(x) = \frac{x-7}{x+1}$**.**



10
5
y
-10
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0
5
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