3.3 Properties of Rational Functions

Definition 1: A *Rational Function* $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$. In fact, the domain of $R(x)$ is such that $D = \{\, x \,|\, Q(x) \neq 0\}$.

Rational functions may seem extremely overwhelming. We will approach this in a similar way, but a little different manner, then other subjects in this book. Usually we break down each concept into a subcategory and then choose the correct manner to solve the problem. With rational functions we have one algorithm that will cover an analysis and graphing of all rational functions.

Before we lay out this step-by-step plan, we must first define and discuss several concepts that pertain to rational functions, the first of which are the vertical and horizontal asymptotes.

Consider the most basic rational function: $f(x) = \frac{1}{x}$,with a domain $D = \{\, x \,|\, x \neq 0\}$.

The graph of $f(x) = \frac{1}{x}$ demonstrates both vertical and horizontal asymptotes.

Definition 2: The line $x = c$ is a *Vertical Asymptote* of $f(x)$ if $f(x) \longmapsto \pm\infty$as x approaches c from either the left or the right.

Notice that the domain of $f(x) = \frac{1}{x}$ is all real numbers except when $x = 0$, which is the vertical line that the function cannot cross. If it crosses this line, then we have division by 0. This is never allowed. As the function approaches $x = 0$ from the left, it will tend toward negative infinity vertically. As the function approaches from the right, it will tend toward positive infinity in the vertically. Here the vertical asymptote is the line $x = c$, where c makes the denominator zero. There is only one exception, which will be discussed later.

Definition 3: The line $y = L$ is a *Horizontal Asymptote* of $f(x)$ if $f(x) \longmapsto L$ as x as approaches $\pm\infty$.

This concept is a little more involved, but it can be broken down into three different categories depending on the degrees of the polynomials of the numerator and the denominator.

Recall the definition of the rational function $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$, and $Q(x)$ are polynomials and that the Degree of the polynomial is the highest power of the polynomial. For example if $f(x) = x^7 + x^3 + 4$ then the degree of $f(x)$ is 7, or $deg(f(x)) = 7$.

Finding the horizontal asymptote is found by racing the denominator and the numerator as $x \longrightarrow \pm\infty$. The existence and/or location of the horizontal asymptote will depend on the dregree of the polynomials.

There are three cases.

(1) If $deg(P(x)) > deg(Q(x))$, then as x goes to infinity, the top will eventually take over and $R(x) = \frac{P(x)}{Q(x)}$ will also go to infinity and will not tend toward any constant line. In this case, there is no horizontal asymptote. However, the end behavior may be predicted by using long division. Whatever remains after $P(x)/Q(x)$ for the exemption of the remainder is known as the oblique asymptote, which is a function of x. This will become much clearer later with an example.

(2) If $deg(P(x)) < deg(Q(x))$, then as x goes to infinity, the denominator will outgrow the numerator and $R(x)$ will tend to 0. In this case the horizontal asymptote is the line $y = 0$.

(3) If $deg(P(x)) = deg(Q(x))$, then as x goes to infinity, the denominator and the numerator will remain in pace with each other. So if $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and $Q(x) = b_n x^n + \cdots + b_1 x + b_0$, then $deg(P(x)) = deg(Q(x)) = n$ and the horizontal asymptote will be $y = \frac{a_n}{b_n}$, the quotient of the coefficients of the highest powers.

There is one last property to discuss before we implement our algorithm for analyzing rational functions. Formally known as removable discontinuity, we will refer to them herein as holes.

Consider the $f(x) = \frac{1}{x}$ and $g(x) = \frac{(x-1)}{x(x-1)}$. Some may be tempted to divided out the term $x - 1$ from $g(x)$, but this would be a mistake as this would change the domain of $g(x)$.

Another mistake is to make the claim that $f(x) = g(x)$, but this is only true when $x \neq 1$.

Consider the domains of each function: $D_f = \{\, x \,|\, x \neq 0\}$ **and** $D_g = \{\, x \,|\, x \neq 0,\, x \neq 1\}$**. Since the domains are not the same, the functions are not the same. Moreover, the factor** $(x-1)$ **exists in both the numerator and the denominator, which create a hole in** $f(x) = \frac{1}{x}$ **at** $x = 1$**, not a vertical asymptote.**

Example 1: Exploit the properties of $f(x) = \frac{1}{x^2-4}$ **to find the graph the function.**

(1) Factor: $f(x) = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$

(2) Find the Domain: $D_f = \{\, x \,|\, x \neq 2,\, x \neq -2\}$

(3) Find the Intercepts:

The *x-intercept* occurs when $y = 0$**. Notice here that** $\frac{1}{x^2-4} \neq 0$**, so no *x-intercept* exists.**

The *y-intercept* occurs when $x = 0$**. Therefore,** $(0, -\frac{1}{4})$ **is the *y-intercept.***

(4) Find the Asymptotes:

The vertical asymptote occurs at the lines $x = -2$ **and** $x = 2$**.**

The Hhorizontal asymptote occurs the line $y = 0$**. This is because the** $deg(1) = 0$ **and** $deg(x^2 - 4) = 2$**. So by Case 2 from above, the horizontal asymptote is the line** $y = 0.$

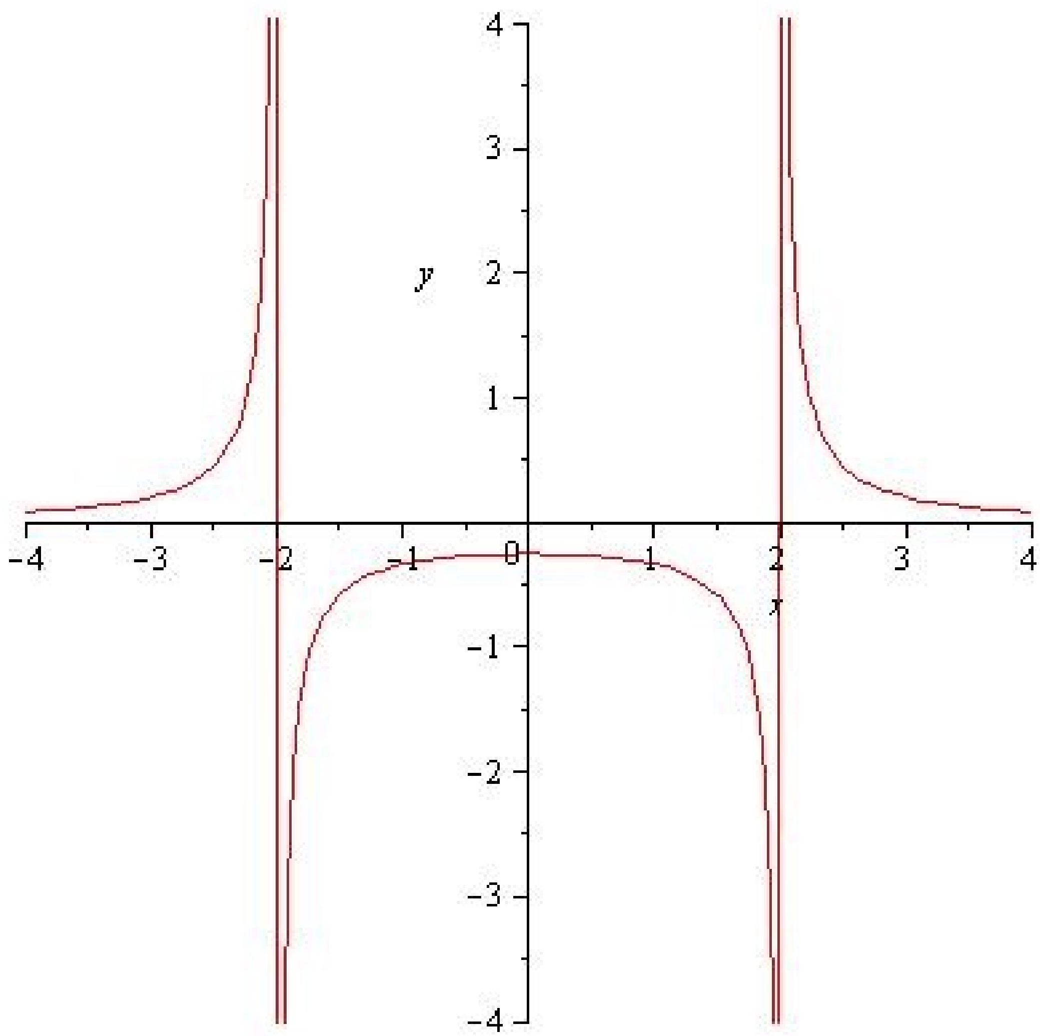
There is no oblique asymptote. Horizontal and oblique asymptotes cannot exist simultaneously.

(5) Find the Holes:

Since there are no common factor in the numerator and the denominator, there are no holes.

Now it is time to use the information gathered about the function to create a graph of $f(x) = \frac{1}{x^2-4}$**.**

These steps will remain the same whenever analyzing and graphing a rational function; however, some properties will change. Notice here there are two vertical vsymptotes and no *x-intercept.* These properties that will influence the shape of the graph. Also notice no information is given of the behavior of the graph outside of the two vertical aymptotes. We need only find the points generated outside the vertical asymptotes. Here $f(-3) = (-3, \frac{1}{5})$ and $f(3) = (3, \frac{1}{5})$. Since there are no *x-intercepts*, the end behavior beyond these asymptotse is easily deatermined.



Example 2: Exploit the properties of $f(x) = \frac{x-1}{x^2-9}$ to find the graph of the function.

(1) Factor: $f(x) = \frac{x-1}{x^2-9} = \frac{x-1}{(x-3)(x+3)}$

(2) Find the Domain: $D_f = \{\, x \,|\, x \neq 3,\, x \neq -3\}$

(3) Find the Intercepts:

The *x-intercept* occurs when $y = 0$. Notice here that $\frac{x-1}{x^2-9} = 0 \Longrightarrow x - 1 = 0 \Longrightarrow x = 1$. So the *x-intercept* exists and occurs at $(1,\, 0)$.

The *y-intercept* occurs when $x = 0$. Therefore, $(0,\, \frac{1}{9})$ is the *y-intercept.*

(4) Find the asymptotes:

The vertical asymptote occurs at the lines $x = -3$ and $x = 3$, as $D_f = \{\, x \,|\, x \neq 3,\, x \neq -3\}$.

The horizontal asymptote occurs at the line $y = 0$. This is because the $deg(x - 1) = 1$ and $deg(x^2 - 9) = 2$. So by Case (2) from above, the horizontal asymptote is the line $y = 0$.

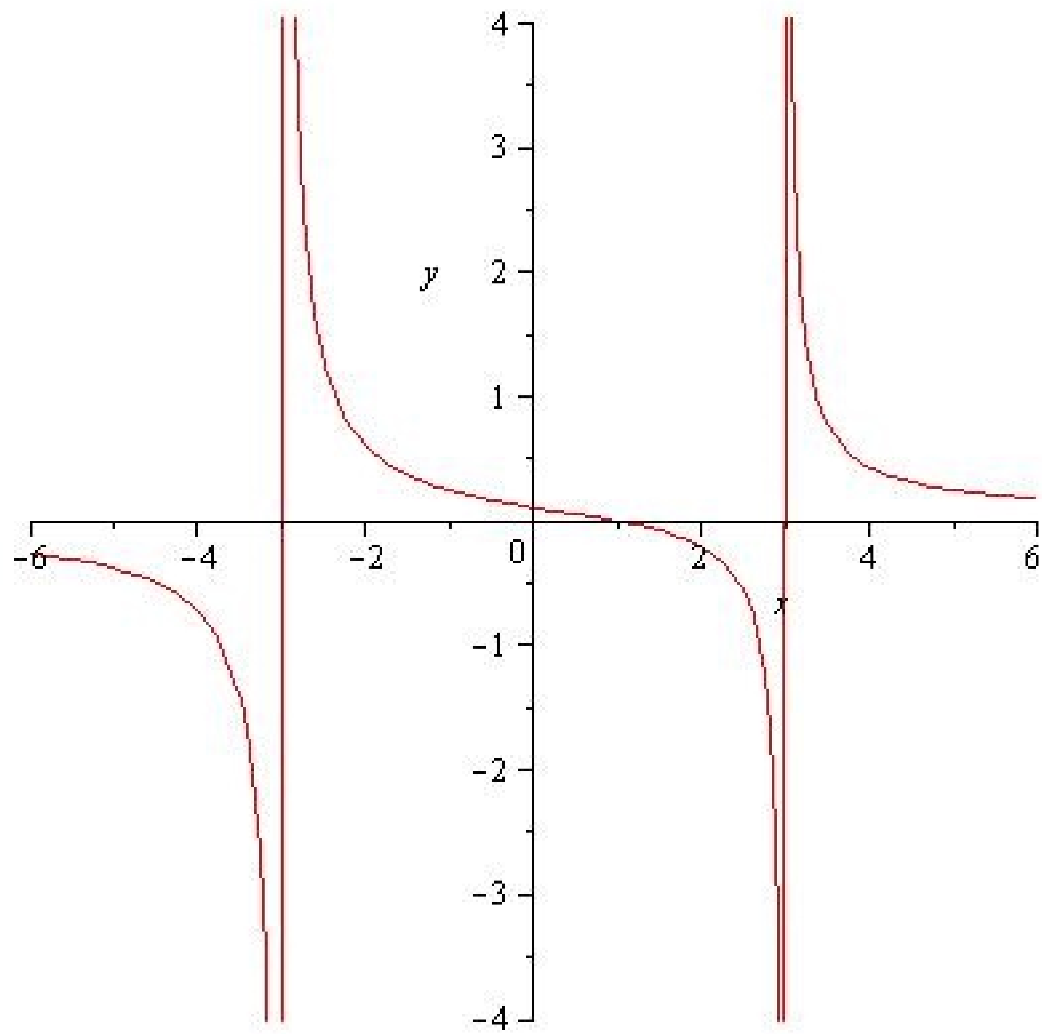
There is no oblique asymptotes. Horizontal and oblique asymptotes cannot exist existing simultaneously.

(5) Find the Holes:

Since there are no common factors in the numerator or denominator, there are no holes.

Now it is time to use the information we have gathered about the function to create a graph of $f(x) = \frac{x-1}{x^2-9}$.

The above questions will remain the same whenever analyzing and graphing a rational function. However, some properties will change and we need to take nate of them. Most important is that we note that there are the vertical asymptotes and no *x-intercept.* These are two of the most important properties that will directly influence the shape of the graph. Also notice no information is given as to the behavior of the graph outside of the two vertical aymptotes. Again, find the points generated by $f(-4) = (-4,\, -\frac{5}{7})$ and $f(4) = (4,\, \frac{3}{7})$ - and see if these will be above or below the *x-axis*, since there is no *x-intercept.*



Example 3: Exploit the properties of $f(x) = \frac{x^2-1}{x^2+x-12}$ to find the graph of the function.

(1) Factor: $f(x) = \frac{x^2-1}{x^2+x-12} = \frac{(x-1)(x+1)}{(x-3)(x+4)}$

(2) Find the Domain: $D_f = \{\, x \,|\, x \neq 3,\, x \neq -4\}$

(3) Find the Intercepts:

The *x-intercept* occurs when $y = 0$. Notice here that $\frac{(x-1)(x+1)}{(x-3)(x+4)} = 0 \Longrightarrow (x-1)(x+1) = 0 \Longrightarrow x = 1, -1$. So the *x-intercept* exists and occur at $(1, 0), (-1, 0)$.

The *y-intercept* occur when $x = 0$. Therefore, $(0, \frac{1}{12})$ is the y-intercept.

(4) Find the Asymptotes:

The vervtical asymptote occurs at the lines $x = -4$ and $x = 3$.

The Horizontal ashmptote occurs the line $y = 1$. This is because the $deg(x^2 - 1) = 2$ and $deg(x^2 + x - 12) = 2$. So by Case 3 from above, we have that the horizontal asymptote is the line $y = 1$.

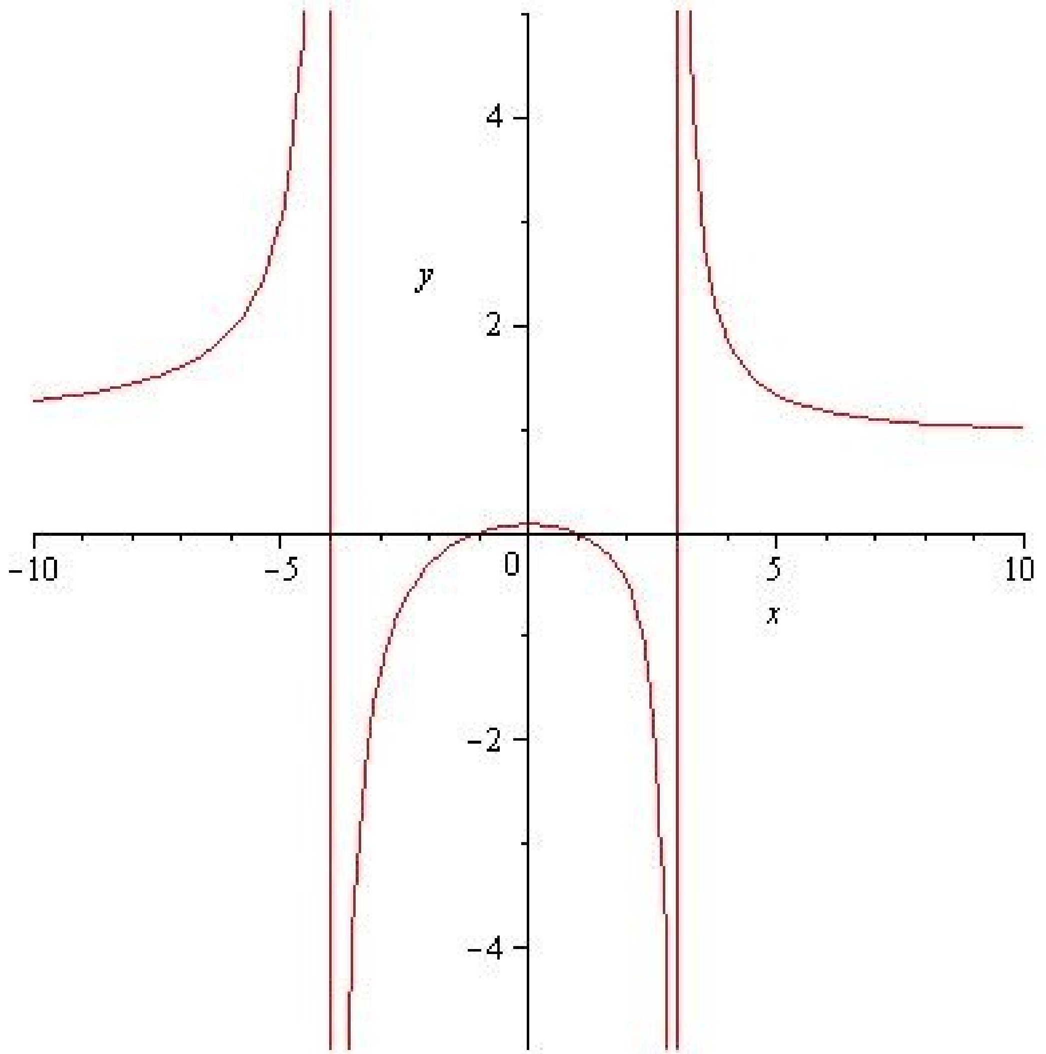
There is no oblique asymptote. Horizontal and oblique asymptote cannot exist simultaneously.

(5) Find the Holes;

Since there are no common factor in the Numerator or Denominator, there are no Holes.

Now it is time to use the information we have gathered about the function to create a graph of $f(x) = \frac{x^2-1}{x^2+x+12}$.

The above questions will remain the same whenever analyzing and graphing a rational function. However, some properties will change and we need to take note of what these are. Most important is that we note that there are two non-symmetric vertical asymptotes and two *x-intercepts*. These are two of the most important properties that will directly influence the shape of the graph. Another important quality to notice about this graph its that we are not given any information in these steps as to the behavior of the graph outside of the two vertical aymptotes. This is actually quite simple. We need only find the points generated by $f(-5) = (-5, 3)$ and $f(5) = (5, \frac{4}{3})$ and see if these will be above the x-axis, since there is no x-intercept.



Example 4: Exploit the properties of $f(x) = \frac{x^2+x-12}{x^2-9}$ to find the graph the function.

(1) Factor: $f(x) = \frac{x^2+x-12}{x^2-9} = \frac{(x+4)(x-3)}{(x-3)(x+3)}$. Now, since we have the factor $(x-3)$ in both the numerator and denominator, we will have a hole at $(3, f(3))$ but the $f(3)$ does not exist.

Since the function does not exist at $(3, f(3))$, we have to make a new function, call it $F(x) = \frac{x+4}{x+3}$. Notice that $f(x) = F(x)$, but only when $x \neq 3$. We can use this $F(x)$ to answer all of the following questions

about the original function except for those invlovong the domain of the original function.

(2) Find the Domain: $D_f = \{\, x \,|\, x \neq 3,\, x \neq -3\}$ **and** $D_F = \{\, x \,|x \neq -3\}$

(3) Find the Intercepts:

The *x-intercept* occurs when $y = 0$**. Notice here that** $F(x) = \frac{x+4}{x+3} = 0 \Longrightarrow x + 4 = 0 \Longrightarrow x = -4$**. So the x-intercept exists and occurs at** $(-4,\, 0)$**.**

The *y-intercept* occurs when $x = 0$**. Therefore,** $(0,\, \frac{4}{3})$ **is the y-intercept.**

(4) Find the Asymptotes:

The vertical asymptote occurs at the line $x = -3$**.**

The horizontal asymptote occurs the line $y = 1$**. This is because the** $deg(x^2 - 9) = 2$ **and** $deg(x^2 + x - 12) = 2$**. Or more simply,** $deg(x + 4) = deg(x + 3)$. **So by Case 3 from above, we have that the horizontal asymptote is the line** $y = 1$.

There is no oblique asymptote, because you cannot have a horizontal and oblique asymptote existing simultaneously.

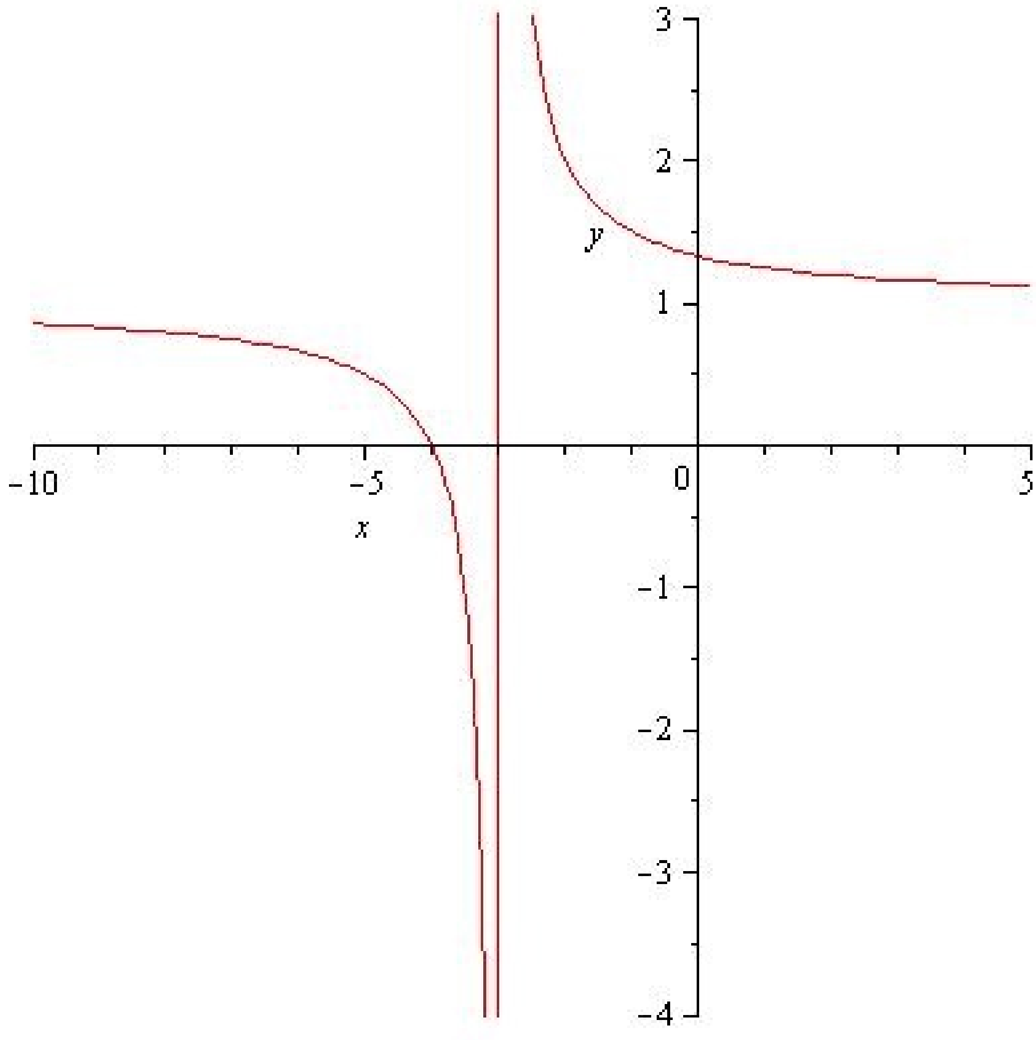
(5) Find the Holes:

Our function $f(x)$ **has a hole, which is a point. We find this point by using** $F(x)$**, since this point does not exist on** $f(x)$**.**

$F(3) = (3,\, \frac{7}{6})$

Now it is time to use the information we have gathered about the function to create a graph of $f(x) = \frac{x^2+x-12}{x^2-9}$**.**

Unlike before, we have all of the information we need to produce the graph.



Example 5: Exploit the properties of $f(x) = \frac{x^2-2x}{x^2+6x+9}$ to find the graph of the function.

(1) Factor: $f(x) = \frac{x^2-2x}{x^2+6x+9} = \frac{(x)(x-2)}{(x+3)(x+3)}$. Now, since we have no common factor in the numerator and denominator, we will not have a hole in this graph. Therefore, we do not need to create a new function either.

(2) Find the Domain: $D_f = \{\, x \,|x \neq -3\}$.

(3) Find the Intercepts:

The *x-intercept* occurs when $y = 0$. Notice here that $f(x) = \frac{x^2-2x}{x^2+6x+9} = \frac{(x)(x-2)}{(x+3)(x+3)} = 0 \Longrightarrow (x)(x-2) = 0 \Longrightarrow x = 0, 2$. So the *x-intercept* exists and occurs at $(0, 0)$ and $(2, 0)$.

The *y-intercept* occurs when $x = 0$. Therefore, $(0, 0)$ is the *y-intercept*, which we found in the last step. Remember we can only have one *y-intercept*. Otherwise, the function will cross a vertical line twice and fail the vertical line test and thereby not qualifying as a function.

(4) Find the Asymptotes:

The vertical asymptote occurs at the line $x = -3$. Does it make a difference that $x = -3$ twice?

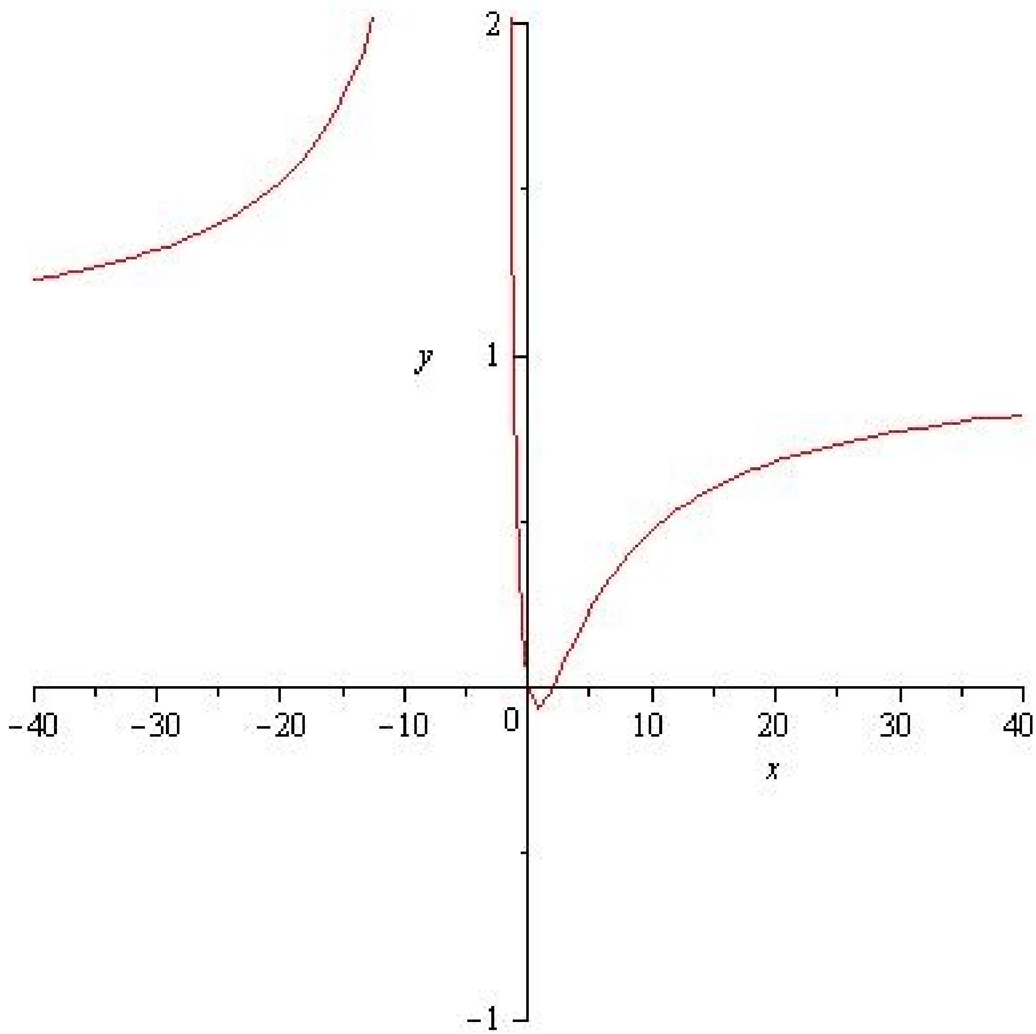
The horizontal asymptote occurs at the line $y = 1$. This is because the $deg(x^2 - 2x) = 2$ and $deg(x^2 + 6x + 9) = 2$. By Case 3, we have that the horizontal asymptote is the line $y = 1$.

There is no oblique asymptote. Horizontal and oblique asymptotes cannot exist simultaneously.

(5) Find the Holes:

Our function $f(x)$ has no holes.

Now use the information we have gathered about the function to create a graph of $f(x) = \frac{x^2+x-12}{x^2-9}$. Notice that the function can cross the horizontal asymptote, as long as we tend towards $y = 1$ as $x \longrightarrow \pm\infty$. However, you may never cross a vertical asymptotes, as this would required division by zero.



Example 6: Exploit the properties of $f(x) = \frac{x^2-4}{x}$ to find the graph of the function.

(1) Factor: $f(x) = \frac{x^2-4}{x} = \frac{(x-2)(x+2)}{x}$. Now, since we have no common factor in the numerator and denominator, we will not have a Hole in this graph. Therefore, we do not need to create a new function.

(2) Find the Domain: $D_f = \{\, x \,|x \neq 0\}$.

(3) Find the Intercepts:

The *x-intercept* occurs when $y = 0$. Notice here that $f(x) = \frac{x^2-4}{x} = \frac{(x-2)(x+2)}{x} = 0 \Longrightarrow (x+2)(x-2) = 0 \Longrightarrow x = -2, 2$. So the *x-intercept* exists and occurs at $(-2, 0)$ and $(2, 0)$.

The *y-intercep*t occurs when $x = 0$. Therefore, the *y-intercep*t does not exist since having $x = 0$ would violate the domain of the function.

(4) Find the Asymptotes:

The vertical asymptote occurs at the line $x = 0$.

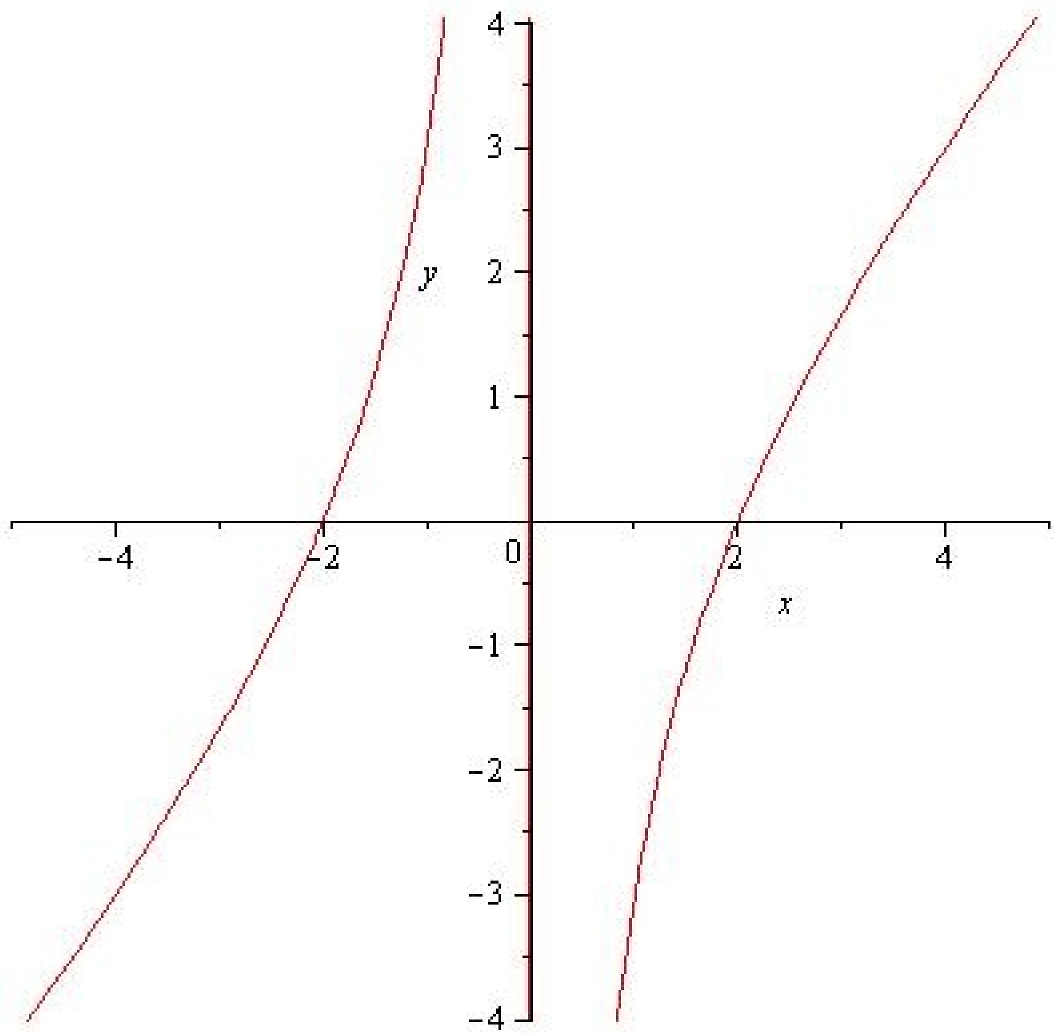
The horizontal asymptote occurs the line $y = x$. This is because the $deg(x^2 - 4) = 2$ and $deg(x) = 1$. By Case (1) we have that the horizontal asymptote does not exist.

The oblique asymptote is the line $y = x$. When we divide the numerator by the denominator we get that $f(x) = x - \frac{4}{x}$.

(5) Find the Holes:

Our function $f(x)$ has no holes.

Now use the information gathered about the function to create a graph of $f(x) = \frac{x^2-4}{x}$.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