3.2 Real Solutions and Polynomial Long Division

In this section we will discuss polynomials of degree greater than two. Quadratics and linear polynomials will play a significant role in solving polynomials of any degree. The greater the degree, the more subroutines will be required and the more time consuming the process will be. Often problems appear difficult at first, but if we practice the methods introduced here with rigor, they inevitably become simple. This is a reoccurring theme of the course, but it is more evident in this section. The key is to practice and not give up. If a student is immediately intimidated by the appearance of the material and decides to quit, then the student has made a poor and deliberate self-defeating choice. If the unfamiliar appearance of a challenge causes a student to quit, then all hope is lost. It does not ever have to be this way in mathematics. Focus on memorizing the rules and practice the examples until you completely understand the mechanics. Students often do well in this section if they have the discipline to practice. Write and re-write the examples and memorize the rules and you will certainly succeed.

Recall from the previous section, if $(x-a)$ is a factor of a polynomial $f(x)$, then $f(a)=0$ and a is known as a zero or a root of $f(x)$. We call $(x-a)$ a factor of $f(x)$ because it divides the polynomial evenly leaving no remainder. It is no different than 4 being a factor of 20, as 20 divided by 4 is 5.

When dividing polynomials by known factors of the polynomial we use polynomial long division. Many prefer synthetic division, but this is not advised. There are many reasons synthetic division is inferior, but the main reason is that you do not gain any mathematical benefit from this practice. Always remember that in mathematics you want to memorize the methods that give you the greatest benefit in solving the widest range of problems and help reinforce basic skills. Polynomial long division will seem daunting at first, but after a few repetitions it will reveal itself as quite simple.

Example 1: Let $f(x)=x^3+x^2-17x+15$. Given the factor $(x-1)$ of $f(x)$, factor $f(x)$ completely.

The goal here is to correctly use polynomial long division to divide a degree one polynomial into a degree three polynomial. The result will be a degree two polynomial, also know as the quadratic. Here

we are given that $x-1$ is a factor of $f(x)$. Later we will define a method for finding linear factors. Notice that we list the polynomials in descending order of degree. If there is a term missing, then we must include that term and multiply it by zero to act as a placeholder. For example, if we were dividing by (or dividing into) a polynomial of the form x^2+1, when using polynomial long division we would write this as $x^2+0\cdot x+1$. This way all powers are represented.

The process of polynomial long division is one method repeated several times. We look at the far-left of the divisor $x-1$ and ask what we need to multiply the far-left term x by to get the far-left term of the dividend x^3. The value that we seek is x^2, so we multiply $x^2(x-1)=x^3-x^2$ and we subtract the two far-left terms of the original $f(x)=x^3+x^2-17x+15$ by x^3-x^2. Do not forget to subtract the entire term. This is a common mistake. Now just repeat this procedure until we reach zero. If there is a remainder, then the divisor $x-1$ is not a factor of $f(x)=x^3+x^2-17x+15$.

$$
\begin{array}{r|llll}
 & x^2 & +2x & -15 & \\ \cline{2-5}
x-1 & x^3 & +x^2 & -17x & +15 \\
 & -(x^3 & -x^2) & & \\ \cline{2-5}
 & & 2x^2 & -17x & +15 \\
 & & -(2x^2 & -2x) & \\ \cline{3-5}
 & & & -15x & +15 \\
 & & & -(-15x & -15) \\ \cline{4-5}
 & & & & 0
\end{array}
$$

Now we can factor the remaining quadratic by asking the usual questions: What two numbers add to give me 2 and multiply to give me -15? The result is 5 and -3. Then $x^2+2x-15=(x+5)(x-3)$. Finally, $f(x)=x^3+x^2-17x+15=(x-1)(x+5)(x-3)$, and we are done.

The question is, how do we find those factors needed to reduce our original function to a quadratic that we can factor? The answer is not direct, but we have a method for finding possible candidates.

Rational Zeros Theorem:

Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n>0$, $a_n\neq 0$, $a_0\neq 0$, is a polynomial with integer coefficients. If $\frac{p}{q}$ is a rational zero of $f(x)$ in lowest terms, then p must divide a_0, and q must divide a_n.

Example 2: Factor $f(x) = 2x^3 + 11x + 7x - 20$ completely.

First we make a list of all possible values of p and q. Note that the leading coefficient is no longer 1, but 2.

Here p must divide -20, so $p = \pm1, \pm2, \pm4, \pm5, \pm10, \pm20$. Note that all values will be both positive and negative no matter the sign of a_o. In this case we have 10 possibilities for p.

Now q must divide 2, so there are only two possibilities: $q = \pm1, \pm2$.

Possible factors are $\frac{p}{q} = \pm1, \pm2, \pm4, \pm5, \pm20, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{4}{2}, \pm\frac{5}{2}, \pm\frac{10}{2}, \pm\frac{20}{2}$. Many of these are duplicates, but it is best to substitute any fractions last.

The degree of $f(x)$ is three, and we can only have as many as three possible rational roots. What is the most efficient way to check these values? We know that if $\frac{p}{q}$ is a rational root, then we must have $f(\frac{p}{q}) = 0$. The most efficient method would be to substitute each possible value into the given function starting with computing the least complicated values first until our result is zero.

$f(-1) = 2(-1)^3 + 11(-1)^2 + 7(-1) - 20 = -2 + 11 - 7 - 20 \neq 0$

$f(1) = 2 + 11 + 7 - 20 = 20 - 20 = 0$

This means that $x = 1$ is a zero, and that $(x - 1)$ divides evenly.

Now we repeat the given algorithm for polynomial long division. Notice that we use $(x-1)$ in both examples. This is not coincidence. The biggest mistake in this procedure is not distributing the -1 in each step. Students tend to add the second term instead of subtracting. It is one of the most common mistakes in all levels of mathematics, especially pre-calculus and calculus.

$$
\begin{array}{r|llll}
 & 2x^2 & +13x & +20 & \\
\hline
x-1 & 2x^3 & +11x^2 & +7x & -20 \\
 & -(2x^3 & -2x^2) & & \\
\cline{2-3}
 & & 13x^2 & +7x & -20 \\
 & & -(13x^2 & -13x) & \\
\cline{3-5}
 & & & 20x & -20 \\
 & & & -(20x & -20) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Now all we need to do is factor the remaining quadratic equation $2x^2 + 13x + 20$. Recall that we are looking for two numbers, those which will add to give 13 and multiply to give 40. Notice that these numbers are 8 and 5. Since the leading coefficient is not 1, we can write $2x^2+13x+20 = 2x^2+8x+5x+20 = 2x(x+4)+5(x+4) = (x+4)(2x+5)$. Therefore, our original $f(x) = 2x^3 + 11x + 7x - 20$ factors completely to $f(x) = (x - 1)(2x + 5)(x + 4)$, and we are done.

Recall that the original polynomial and the polynomial factor need to be in descending order of powers for this method to work. What if we are missing a power? Consider the following.

Example 3:) Factor $f(x) = x^3 - 7x - 6$ completely.

Notice there is no x^2 term. There is an easy way to remedy this. Write $f(x) = x^3 + 0x^2 - 7x - 6$.

We make our list of p's and q's as before.

Here $p = \pm1, \pm2, \pm3, \pm6$, and again $q = \pm1$. Then $\frac{p}{q} = \pm1, \pm2, \pm3, \pm6$.

$f(-1) = (-1)^3 - 7(-1) - 6 = 0$ and we have a linear factor we can divide by, which is $(x + 1)$.

$$
\begin{array}{r|rrrr}
 & x^2 & -x & -6 & \\
\hline
(x+1) & x^3 & +0x^2 & -7x & -6 \\
 & -(x^3 & +x^2) & & \\
\cline{2-3}
 & & -x^2 & -7x & -6 \\
 & & -(x^2 & -x) & \\
\cline{3-5}
 & & & -6x & -6 \\
 & & & -(-6x & -6) \\
\cline{4-5}
 & & & & \mathbf{0}
\end{array}
$$

As in the previous example, once we have a quadratic we my use whatever means necessary to factor the remaining polynomial.

The simplest way to do this is ask; what two numbers multiply to give -6 and add to give -1. The answers are -3 and 2.

We can now write $f(x) = x^3 - 7x - 6 = (x + 1)(x - 3)(x + 2)$.

Notice that all of our examples are degree three polynomials. If they were degree four, we would just repeat the process above until we were left with a quadratic or an irreducible polynomial.