3.1 Properties of Polynomial Functions

Definition 1: A *Polynomial Function* (in one variable, which is always assumed for the scope of this course) is a function that is the sum (or difference) of terms written in the general form as $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_o$, where the a_i are constants for $i = 0, 1, \cdots (n-1), n$, each of which indicates an arbitrary constant in the real numbers and some may be zero. The variables x^i, have all non-negative integer powers, that is the same values of i , which stands for iteration. These may also be zero. The largest power of x is known as the *Degree* (or *Order*) of the polynomial.

As with any definition contained in this book, it will become more clear as we progress throughout the section and are exposed to more and more examples. Nevertheless, the definition of any mathematical concept is written in stone and must always be adhered to and thus memorized.

Example 1: According to the Definition of a Polynomial, which of the following are polynomials?

(a) $f(x) = 7x^3 + 5x + 1$

Yes, $f(x)$ is a polynomial as all of the coefficients are real numbers and all of the powers of x are non-negative integers.

(b) $g(x) = \sqrt[3]{x^4}$

No, $g(x) = x^{\frac{4}{3}}$ and $\frac{4}{3}$ is not an integer. Therefore, $g(x)$ is not a polynomial.

(c) $h(x) = \frac{x^5+x^3+x+1}{x-1}$

No, $h(x)$ is known as a Rational function. The numerator is a polynomial and the denominator is a polynomial, but one over the other does not make a polynomial.

(d) $i(x) = 5$

Yes, $i(x)$ is a constant and a constant,and a real constant is a polynomial. Here the power of x is zero, which equals one. That is, $i(x) = 5x^0 = 5$.

(e) $j(x) = x^2(x-1)(x^3+5)$**.**

Each factor x^2, $(x-1)$, (x^3+5) are polynomials, so the product of these creates another polynomial.

Definition 2: If $f(x)$ is a function and $f(a) = 0$, then a is a *Real Zero* of $f(x)$ if a is a real number.

As a result of Definition 2, the following statements are equivalent (TFAE).

(1) a is a solution to the equation $f(a) = 0$.

(2) The *x-intercept* is the point $(a,\ 0)$ on the graph of $f(x)$.

(3) The term $(x-a)$ divides $f(x)$ evenly and leaves no remainder. That is, $(x-a)$ is known as a *factor* of $f(x)$.

Definition 3: The *Degree* of a polynomial is the highest power of x in the polynomial. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_o$, then $f(x)$ is a *degree n polynomial.* This may be written as $deg(f(x)) = n$.

The degree can tell you many things about the function of a polynomial. For example, the polynomial will have at most n roots when the multiplicity is n, and at most $n-1$ turning points.

Definition 4: The *multiplicity m* of a certain zero a of a function $f(x)$, where *m* is the largest number such that $(x-a)^m$ is a factor of $f(x)$.

The importance of the multiplicity is whether or not it is even or odd. You must remember this following.

When the multiplicity is and even number, then the function will only touch the *x-axis* at $x=a$. As a result, the positivity or negativity of the function *will not change.*

When the multiplicity is odd then the function will only Cross at $x=a$. As a result, the positivity or negativity of the function will change.

Definition 5: A polynomial function $f(x)$ is called *even* when $f(-x)=f(x)$.

Definition 6: A polynomial function $f(x)$ is called *odd* when $f(-x)=-f(x)$.

Definition 5 can be illustrated with $f(x)=x^2$. This is an even function because $f(-x)=(-x)^2=x^2=f(x)$.

Definition 6 can be illustrated with $f(x)=x^3$. This is an odd function because $f(-x)=(-x)^3=-x^3=-f(x)$.

These definitions are important when talking about the End Behavior of a function. We will talk about End Behavior after we discuss some examples of polynomial graphs.

Example 2: Analyze the polynomial function $f(x)=(x-1)(x+3)^2(x+1)$.

(a) First find the *x-intercept* of $f(x)$.

Repeat the mantra "the *x-intercept* occurs when $y=0$.".

If we let $(x-1)(x+3)^2(x+1)=0$, then $x=1,-3,-1$.

These are the roots. The *x-intercept* is actually a point so we have that $(1,\ 0),(-3,\ 0),(-1,\ 0)$.

(b) Now find the y-intercepts of $f(x)$.

Repeat the mantra "the y-intercept occurs when $x = 0$."

If $f(0) = (-1)(3)^2(1) = -9$,

then *y-intercept* is $(0, -9)$.

Notice that there is only one *y-intercept* point. This is because $f(x)$ is a function. If there were more than one, then $f(x)$ would violate the *Horizontal Line Test* and, by definition, not be a function.

Notice that when distributed, $f(x) = x^4 + 6x^3 + 8x^2 - 6x - 9$, so the degree of $f(x)$is four. You could also determine the degree in the factored form by adding up the degrees of each individual factor. Nevertheless, this means that as $f(x)$ will behave like the function $h(x) = x^4$as $f(x)$ approaches $-\infty$and $+\infty$, because then the degree of the function is the dominant force of the function. This is what we refer to as the *End Behavior* of the function.

Example 3: Analyze the polynomial function $f(x) = (x+2)^2(x-2)^2(3x+1)$.

(a) First find the *x-intercept* of $f(x)$.

Repeat the mantra "the *x-intercept* occurs when $y = 0$."

If we let $(x + 2)^2(x - 2)^2(3x + 1) = 0$, then $x = -4,\ 2, -\frac{1}{3}$.

These are the roots. The *x-intercept* is actually a point so we have that $(-4,\ 0), (2,\ 0), (-\frac{1}{3},\ 0)$.

(b) Now find the y-intercepts of $f(x)$.

Repeat the mantra "the *y-intercept* occurs when $x = 0$."

If $f(0) = (4)(-2)^2(1) = 16$,

the *y-intercept* is $(0,\ 16)$.

Notice that there is only one point. There can be only one y-intercept, because $f(x)$ is a function. If there were more than one, then $f(x)$ would violate the *HLT* and, by definition, would not be a function.

Notice that when distributed, $f(x) = 3x^5 + 11x^4 - 40x^3 - 84x^2 + 224x - 64$. The degree of $f(x)$ is *five*. The degree also determines the degree, in the factored form, by adding up the degrees of each individual factor. Nevertheless, this means that $f(x)$ will behave like the function $h(x) = x^5$ as $f(x)$ approaches $-\infty$ and $+\infty$, because then the degree of the function is the dominant force of the function. This is what we refer to as the *end behavior* of the function.