2.4 One-to-One and Inverse Functions

Defintition1: A Function $f(x)$ is *One-to-One* if two unique inputs from the domain of $f(x)$, say $x_1 \neq x_2$ in D_f , implies that $f(x_1) \neq f(x_2)$ in R_f, the range of $f(x)$. Equivalently, $f(x)$ is One-to-One if $f(x_1) = f(x_2)$ directly implies that $x_1 = x_2$.

A One-to-One function is abbreviated as 1-1.

Recall that a given curve is a function if it passes the Vertical Line Test, which says that a curve is a function if and only if any vertical line that passes through the curve does not pass through the curve more than once. That is the same as saying that a single element from the domain of a function can not yield more than one element in the range (set of all outputs.) As a reminder, let us restate the definition of a function.

Definition 2: Suppose A and B are two non-empty sets. A *Function* assigns to every element of the set A exactly one element of the set B.

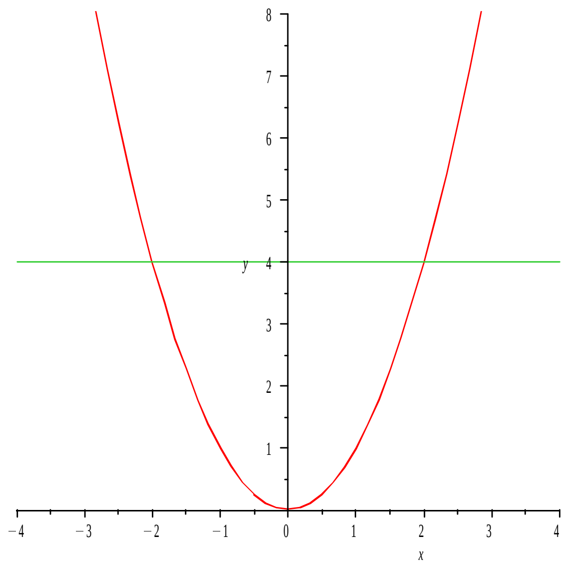
There is a visual way to interpret Definition 2. Essentially the definition of a function is saying is that for each x_1 in the Domain of some function $f(x)$ (here we call it set A) there is exactly one single $f(x_1)$ in the Range of this function (set B.) The visual interpretation of this concept is called the *Vertical Line Test* or *V.L.T.*.

Vertical Line Test: A curve $y = f(x)$ in the Cartesian Plane is a Function if and only if any vertical line that passes through that curve intersects the curve in at most one point.

Like wise, there is a visual way to interpret Definition 1 and the 1-1 qualification of a function. This is known as the *Horizontal Line Test*, or *H.L.T.*.

Horizontal Line Test: A function $y = f(x)$ in the Cartesian Plane is a 1-1 function if and only if any horizontal line that passes through that curve intersects the curve in at most one point.

The perfect example of a function that is not 1-1 is $f(x) = x^2$. Notice that if $x_1 = -2$ and $x_2 = 2$, then $f(x_1) = f(x_2) \Longrightarrow f(-2) = f(2) \Longrightarrow 4$. Two different values of the independent variable give us the same value in the range R_f, in particular $y = 4$. If we were to draw a straight horizontal line through $y = 4$, the line would cross through the points $(-2,\, 4)$ and $(2,\, 4)$, thus failing the Horizontal Line Test.



Definition 3: Let $f(x)$ be a 1-1 function. Then there exists a function known as the *Inverse Function* of $f(x)$ written as $f^{-1}(x)$, such that $f\left(f^{-1}(x)\right) = x$ for every x in the domain of $f^{-1}(x)$ and $f^{-1}(f(x)) = x$ for every x in the domain of $f(x)$.

Theorem 1: If a function that is always increasing (or always decreasing) on a given interval, then that function will be 1-1 on that interval.

If a function is 1-1, then it has an inverse. A function that is composed with it's inverse will take you back to where you started from, which in this case is x.

Keep repeating the mantra "the Inverse Function switches x and y." The Inverse Function simply switches the inputs with the outputs. In fact, if $y = f(x)$ is 1-1 then the Inverse Function $y = f^{-1}(x)$ is a reflection of $f(x)$ about the line $y = x$. You are simply switching inputs and outputs, x and y's, and vice-versa. That is, $(a,\, b) \longrightarrow (b,\, a)$

as a result of applying the Inverse Function. Apply it again and you go back right to where you started from $(b,\, a) \longrightarrow (a,\, b)$

Also note that the Domain of $f(x)$ **is the Range of** $f^{-1}(x)$ **, and the Domain of** $f^{-1}(x)$ **is the Range of** $f(x)$**. Imagine that the original Function takes you from set A to set B. The Inverse Function takes you from set B to set A.**

We must remember the goal of this course is to prepare us for Calculus. In Calculus we use certain techniques that can be practiced and made ready for future use. For example, finding the inverse of a 1-1 Rational Function will implement similar moves as those used when performing Implicit Differentiation in Calculus. This method is usually considered very challenging. Practicing these next examples will make future methods less challenging, and not exclusively when finding the Inverse Function.

Example 1: Given the 1-1 linear function $f(x) = 5x + 2$**, find** $f^{-1}(x)$ **and the Domain and Range of both functions.**

Remember that we only need to switch x **and** y**, and the solve for** y**.This new** y **will be our** $f^{-1}(x)$**. Since we currently do not have a** y**, let** $y = 5x + 2$ **and we may begin.**

$y = 5x + 2$

$x = 5y + 2$

$x - 2 = 5y$

$\frac{x-2}{5} = y$

$f^{-1}(x) = \frac{x-2}{5}$

The Domain of $f(x)$ **is clearly all real numbers, since it is a line and lines cover the entire length of the *x-axis*. To find the Range of** $f(x)$**, we only need the Domain of** $f^{-1}(x) = \frac{1}{5}(x - 2)$**. Since the Inverse Function is also a line, the Domain is all real numbers. Therefore, the Domain and Range of** $f(x)$ **and** $f^{-1}(x)$ **are both the set of all real numbers.**

By the definition of the Inverse Function, the composition of a function and it's inverse will take you back to the original input$x$**. That is,** $f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$**. We can check our work from Example 1 using this fact.**

$$f\left(f^{-1}(x)\right) = f(\frac{x-2}{5}) = 5(\frac{x-2}{5}) + 2 = x - 2 + 2 = x$$

$$f^{-1}(f(x)) = f^{-1}(5x+2) = \frac{(5x+2)-2}{5} = x$$

Example 2: Given the 1-1 Rational Function $f(x) = \dfrac{2x+3}{x+5}$**, find** $f^{-1}(x)$ **and the Domain and Range of both.**

First let

$$y = \frac{2x+3}{x+5}$$

Now switch x **and** y**.**

$$x = \frac{2y+3}{y+5}$$

Then solve for y**.**

$$x(y+5) = 2y+3$$

Distribute the x**.**

$$xy + x5 = 2y + 3$$

Now bring every term with a y **to the left side and everything else move to the right.**

$$xy - 2y = 3 - 5x$$

Factor the y **out of the left side.**

$$y(x-2) = 3 - 5x$$

Solve for y**.**

$y = \dfrac{3 - 5x}{x - 2}$

The new y is now $f^{-1}(x)$.

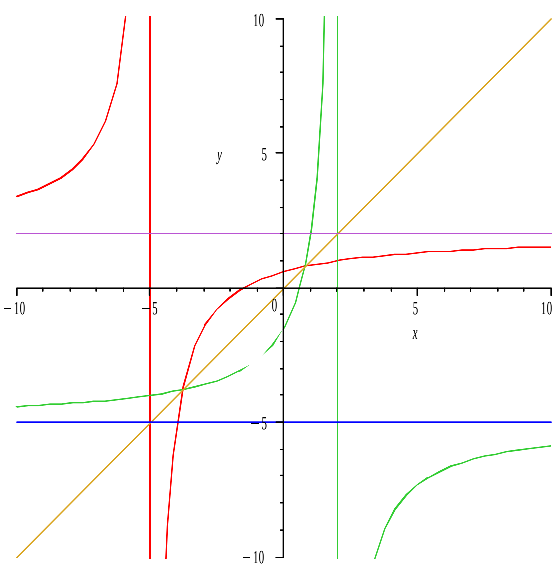
$f^{-1}(x) = \dfrac{3 - 5x}{x - 2}$

The Domain of $f(x)$ is determined by $x + 5 = 0 \Longrightarrow x = -5$, so $D_f = \{ x \mid x \neq -5 \}$. The Range of $f^{-1}(x)$ is the same as the Domain of $f(x)$, but must be written in terms of y, as follows $R_{f^{-1}} = \{ y \mid y \neq -5 \}$.

The Domain of $f^{-1}(x)$ is determined by $x - 2 = 0 \Longrightarrow x = 2$, so $D_{f^{-1}} = \{ x \mid x \neq 2 \}$. The Range of $f(x)$ is the same as the Domain of $f^{-1}(x)$, but must be written in terms of y, as follows $R_f = \{ y \mid y \neq 2 \}$.

The logic for this is simple. Remember, "the Inverse Function switches x and y." The result is a reflection about the line $y = x$.

In the image below, $f(x)$ is red, $f^{-1}(x)$ is green, $y = x$ is yellow, and $y = -5, -2$ is blue and purple, respectively.



By the definition of the Inverse Function the composition of a function and it's inverse will take you back to x. That is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. We can check our results above using this fact.

$$f(f^{-1}(x)) = f(\frac{3-5x}{x-2}) = \frac{2(\frac{3-5x}{x-2})+3}{(\frac{3-5x}{x-2})+5} = \frac{\frac{6-10x}{x-2}+3(\frac{x-2}{x-2})}{\frac{3-5x}{x-2}+5(\frac{x-2}{x-2})} = \frac{\frac{6-10x+3x-6}{x-2}}{\frac{3-5x+5x-10}{x-2}} =$$

$$\frac{-7x(x-2)}{-7(x-2)} = x$$

$$f^{-1}(f(x)) = f^{-1}(\frac{2x+3}{x+5}) = \frac{3-5(\frac{2x+3}{x+5})}{\frac{2x+3}{x+5}-2} = \frac{3(\frac{x+5}{x+5})-5(\frac{2x+3}{x+5})}{\frac{2x+3}{x+5}-2(\frac{x+5}{x+5})} = \frac{\frac{3x+15-10x-15}{x+5}}{\frac{2x+3-2x-10}{x+5}} =$$

$$\frac{-7x(x+5)}{-7(x+5)} = x$$

This confirms that the two are Inverse Functions of one-another.

Since the basic steps of finding an inverse function are almost always the same, we do not need too many examples. Example 1 is the more simple example of finding the inverse function. That is, when the original function is a line, the inverse function will also be a line. Example 2 is a Rational Function consisting one linear function over another. This is a more in depth, and a more useful example of finding the inverse. It would be beneficial if we did another example and we will also need to discuss the Transformation of Asymptotes, which we will discuss in greater detail later.

Example 3: Given the 1-1 Rational Function $f(x) = \frac{2x+7}{x+4}$, find $f^{-1}(x)$ and the Domain and Range of both. Discuss the Horizontal and Vertical Asymptotes as well.

We will follow the same algorithm as the previous example, but without the narrative.

$$f(x) = \frac{2x+7}{x+4}$$

$$y = \frac{2x+7}{x+4}$$

$x = \dfrac{2y+7}{y+4}$

$x(y+4) = 2y+7$

$xy + 4x = 2y + 7$

$xy - 2y = -4x + 7$

$y(x-2) = -4x + 7$

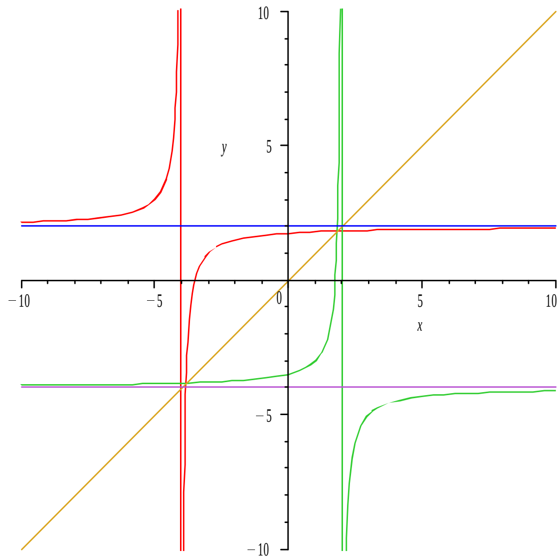
$y = \dfrac{-4x+7}{x-2}$

$f^{-1}(x) = \dfrac{-4x+7}{x-2}$

$D_f = \{\, x \mid x \neq -4 \,\} = R_{f^{-1}} \{\, y \mid y \neq -4 \,\}$

$D_{f^{-1}} = \{\, x \mid x \neq 2 \,\} = R_f \{\, y \mid y \neq 2 \,\}$

Now aside from one very specific anomaly, which we will discuss later, the Vertical Asymptote occurs at the x value that makes the denominator zero. So$f(x)$ has a Vertical Asymptote when $x = -4$. Determining the Horizontal Asymptote is a little more involved (and will be discussed in more detail later,) but basically when the degree of the polynomial in the numerator and the denominator are the same, the Horizontal Asymptote is the coefficient of x in the numerator over that of the denominator. Here for $f(x)$ the Horizontal Asymptote is $y = 2$. Since the inverse is simply the switching of x and y, the Vertical and Horizontal Asymptotes of $f^{-1}(x)$are $x = 2$ and $y = -4$, respectively. In the image below, $f(x)$ is red, $f^{-1}(x)$ is green, $y = x$ is yellow, and $y = 2$, -4 is blue and purple, respectively.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5
y
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