2.3 Composite Functions

Definition 1: Given the functions $h(x)$, $f(x)$, $g(x)$, suppose that$h(x) = f(g(x))$. Here $h(x)$ is known as a *Composite Function*, being such that $f(x)$ is composed of $g(x)$. The Domain of $h(x)$ is the set of all inputs of $g(x)$ such that $g(x)$ is in the set of all inputs of $f(x)$. Composite Functions are sometimes called "functions of functions."

Note that in the above definition, we must first have x in the domain of $g(x)$ and then $g(x)$ in the domain of $f(x)$, in order for x to be in the domain of $h(x)$. Think about it as passing through two locked gates. First you must successfully pass through $g(x)$, then you must pass through $f(x)$.

Example 1: Suppose that $f(x) = x^2 + 4x + 4$ and $g(x) = 3x + 1$.

(a) Find $f(g(x))$ and the domain of the composite.

Remember that whenever we substitute any thing into a function we must use parenthesis.

Simply put, $f(\quad) = (\quad)^2 + 4(\quad) + 4$.

Since $g(x) = 3x + 1$ we have $f(g(x)) = f(3x + 1) = (3x + 1)^2 + 4(3x + 1) + 4$.

It is best to NOT simplify a composite function. When one or either of the functions has a restricted domain the appearance of this restriction may become washed out of the simplified form. Simplification is a misleading concept, as it is a subjective term with no formal definition in mathematics. Find the composite as above and leave it. Especially when attempting to find the domain.

Since both are polynomials, they both have non-restricted domains. That is, their domains are both the set of all real numbers. When we put these together, the domain of $f(g(x))$ is the set of all real numbers.

(b) Find $g(f(x))$and the Domain of the Composite.

Now $g(\quad) = 3(\quad) + 1$.

Since $f(x) = x^2 + 4x + 4$,

$g(f(x)) = g(x^2 + 4x + 4) = 3(x^2 + 4x + 4) + 1$

Again, both functions have the domain of all real numbers. Hence the composite will have the domain of all real numbers.

Example 2: Suppose that $f(x) = x^2 - 5$ **and** $g(x) = \sqrt{x+2}$

(a) Find $f(g(x))$ **and the domain of the composite function.**

Fill in the blanks as before. $f(g(x)) = f(\sqrt{x+2}) = (\sqrt{x+2})^2 - 5$

Again, instinct might have us simplify and then find the domain. This would be a mistake.

If we were to "simplify" we would have

$f(g(x)) = f(\sqrt{x+2}) = (\sqrt{x+2})^2 - 5 = x + 2 - 5 = x - 3$.

This is a linear polynomial with the domain of all real numbers. Herein lies one of the many dangers of simplification.

The domain is in fact not all real number as a consequence of the very definition of a composite function, since the innermost function has a restricted domain. The correct domain is then washed out of the simplified composite Function.

We now know that when we simplify a composite function, we are in jeopardy of changing the domain.

Thus $f(g(x)) = f(\sqrt{x+2}) = (\sqrt{x+2})^2 - 5$ **is fine.**

First we consider the domain of $g(x) = \sqrt{x+2}$**. The function is an even root, so everything inside of the root must be greater than or equal to zero. That is** $x + 2 \geq 0 \Longrightarrow x \geq -2$.

Now consider the domain of the function $f(x)$**. Because** $f(x) = x^2 - 5$ **is a polynomial, it has a domain of all real numbers.**

Therefore the domain of $D_{f(g(x))} = D_{g(x)} = \{\, x \mid x \geq -2 \,\}$.

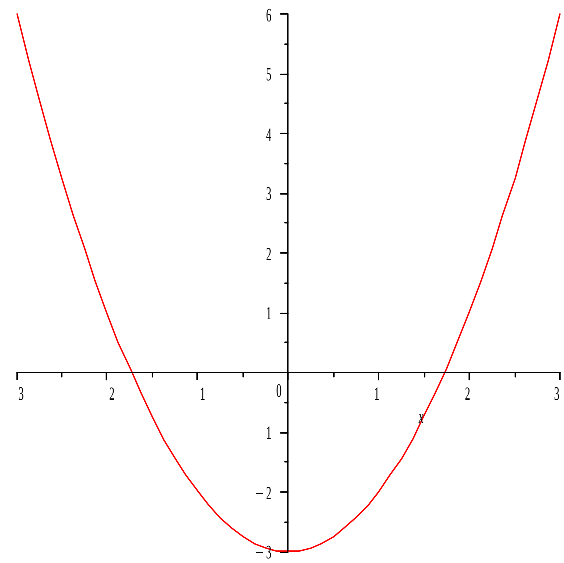
In interval notation, this would be written as $[-2, \infty)$.

(b) Find $g(f(x))$ **and the domain of the composite function.**

Here $g(f(x)) = g(x^2 - 5) = \sqrt{(x^2 - 5) + 2} = \sqrt{x^2 - 3}$.

The last step is valid and not dangerous, because it does not change the domain of the composite.

Now solve $x^2 - 3 \geq 0$. What do we know? We know what the graph of $x^2 - 3$ looks like from the previous section. It is the parabola $y = x^2$ dropped down three units in the vertical. The function $x^2 - 3$ factors over the irrationals to give $(x - \sqrt{3}) \cdot (x + \sqrt{3})$. Now set the factors equal to zero to find the roots $x = \pm\sqrt{3}$. Recall that the roots occur when the function crosses the ***x-axis*** and the quadratic changes from positive to negative, of vice-versa. Since we are trying to find those values of x where $x^2 - 3 \geq 0$, we consider the graph of $x^2 - 3$.



The inequality $x^2 - 3 \geq 0$ is true when $x \geq \sqrt{3}$ and $x \leq -\sqrt{3}$.

Written in interval notation the domain is $(-\infty, -\sqrt{3}) \bigcup (\sqrt{3}, \infty)$.

In set notation $D_{g(f(x))} = \{\, x \mid x \geq \sqrt{3} \, and \, x \leq -\sqrt{3} \}$.

Example 3: Suppose that $f(x) = \dfrac{1}{x+5}$ and $g(x) = \dfrac{1}{x}$.

(a) Find $f(g(x))$and the domain of the composite function.

There is no need to change the method used in the previous two examples.

Use parenthesis and substitute carefully.

$f(g(x)) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}+5}$

Again, it is tempting to simplify. But this is a composite function consisting of two functions with restricted domains.

It is clear that $x \neq 0$, as then we would be dividing by 0 and violating the domain of $g(x)$.

Next, notice that $\frac{1}{x}+5 \neq 0$, as again we would be dividing by zero and violating the domain of the composite function $f(g(x)) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}+5}$.

Now solve the equation $\frac{1}{x}+5 = 0$ for x and kick this number out of the domain of the composite, as well as $x = 0$.

Therefore, $\frac{1}{x} = -5 \implies x = -\frac{1}{5}$ and the domain of the composite function $f(g(x))$ is $D_{f(g(x))} = \{\, x \mid x \neq 0, \, -\frac{1}{5} \}$

If we simplify $f(g(x))$ to get $\dfrac{1}{\dfrac{1}{x}+5} = \dfrac{1}{\dfrac{1}{x}+\dfrac{5x}{x}} = \dfrac{1}{\dfrac{1+5x}{x}} = \dfrac{x}{1+5x}$.

The domain of this result would be $x \neq -\frac{1}{5}$. This is not just more work but also incomplete. It does not have the same domain of the composite that we just found, and therefore it can not be the same function. Two functions can not be the same if they have different domains. This is like claiming that two people are identical twins, even though they have different birthdays or different parents. Simplification is quite often a counterproductive waste of time.

(b) Find $g(f(x))$ and the domain of the composite function.

There is no need to change the method used in the previous two examples. Be sure to use parenthesis and substitute carefully.

$g(f(x)) = \dfrac{1}{\dfrac{1}{x+5}}$

At first glance it appears that $g(f(x)) = x+5$, but this is NOT the case. Remember that this is a composite function. It has a deep structure and we must first get through $f(x)$ before we can move on to $g(x)$. In $f(x) = \frac{1}{x+5}$, $x \neq -5$. Now the denominator of $g(f(x))$ is $\frac{1}{x+5}$ and this will never equal zero. That is, there is no such x that can make $\frac{1}{x+5} = 0$.

Hence, the domain of $g(f(x))$ is simple $D_{g(f(x))} = \{\, x \mid x \neq -5\}$. And we are done.

Example 4: Suppose that $f(x) = \frac{2x+4}{x-3}$ and $g(x) = \frac{5x-10}{7x+3}$

(a) Find $f(g(x))$ and the domain of the composite function.

$$f(g(x)) = f\left(\frac{5x-10}{7x+3}\right) = \frac{2\left(\frac{5x-10}{7x+3}\right)+4}{\left(\frac{5x-10}{7x+3}\right)-3}$$

Do not give in to the urge to simplify. Imagine how much extra work that would be and all done in order to get the wrong solution.

Notice if $7x+3=0$ in either the first term of the numerator or that of the denominator, we would have division by zero. Let us be very clear that this is never allowed. It would also violate the domain of $g(x)$.

First solve $7x+3=0 \Longrightarrow 7x=-3 \Longrightarrow x=-\frac{3}{7}$.

Therefore, $x \neq -\frac{3}{7}$, which is by no coincidence is this the the domain of $g(x)$. Recall that we must first pass through the domain of the innermost function.

Next look at the ultimate denominator and note that $\left(\frac{5x-10}{7x+3}\right)-3 \neq 0$.

$$\left(\frac{5x-10}{7x+3}\right)-3=0 \Longrightarrow \qquad \frac{5x-10}{7x+3}-3=3 \Longrightarrow 5x-10=3(7x+3).$$

$$\Longrightarrow 5x-10=21x+9 \Longrightarrow 5x-10-21x-9=0 \Longrightarrow -16x-19=0 \Longrightarrow x=-\frac{19}{16}$$

The domain of $f(g(x))$ is $D_{f(g(x))} = \left\{ x \mid x \neq -\frac{3}{7}, -\frac{19}{16} \right\}$

(b) Find $g(f(x))$ and the domain of the composite function.

$$g(f(x)) = g(\frac{2x+4}{x-3}) = \frac{5(\frac{2x+4}{x-3}) - 10}{7(\frac{2x+4}{x-3}) + 3}$$

If $x - 3 = 0$ then we have a serious problem in both the first term of the denominator and the first term of the numerator.

So $x \neq 3$, which is also the domain of $f(x)$.

Next $7(\frac{2x+4}{x-3}) + 3 \neq 0$, otherwise the entire denominator is zero.

Again, solve the following for x and kick that number out of the domain.

$7(\frac{2x+4}{x-3})+3 = 0 \Longrightarrow 7(\frac{2x+4}{x-3}) = -3 \Longrightarrow 7(2x+4) = -3(x-3) \Longrightarrow 14x+28 = -3x + 9$

$\Longrightarrow 14x + 28 + 3x - 9 = 0 \Longrightarrow 17x + 19 = 0 \Longrightarrow 17x = -19 \Longrightarrow x = -\frac{19}{17}$

The domain of $g(f(x))$ is $D_{g(f(x))} = \left\{ x \mid x \neq -3, -\frac{19}{17} \right\}$.

If the student rehearses these examples several times (at least five,) then confidence in our skills will grow.

Remember to watch out for the possibility of division by zero, or placing negative numbers inside of an even root. When dealing with a composite function, do NOT simplify. Simplification often results in a loss of critical information.